Miscellany

1. If $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$, write down (i) $3\mathbf{M}$, (ii) $\det\{\mathbf{M}\}$, (iii) $\det\{3\mathbf{M}\}$.

   Show that, in general, $\det\{f\mathbf{M}\} = f^n \det\{\mathbf{M}\}$ where f is a factor, and $\mathbf{M}$ is an $n \times n$ matrix.

2. Find the eigenvalues and eigenvectors of (i) $\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$ and (ii) $\begin{pmatrix} 5 & -7 \\ 1 & -3 \end{pmatrix}$.

   In one case, the eigenvectors are orthogonal. How is this fact related to the structure of the matrix?

3. If $\mathbf{A} = \begin{pmatrix} 5 & -7 \\ 1 & -3 \end{pmatrix}$ (see question 2(ii)), deduce the eigenvalues of $\mathbf{A}^2$.

4. Find the eigenvalues and eigenvectors of

   (i) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 27 \end{pmatrix}$, (ii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, (iii) $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}$.

5. If $z_1 = 2 + 2i$ and $z_2 = -1 + 3i$, deduce the modulus and argument (in degrees) of

   (i) z_1 (ii) z_2 (iii) z_1^* (iv) $z_1 z_2$ (v) z_2 / z_1 (vi) z_1^* / z_2.

6. For the same values of z_1 and z_2 as in question 5, find

   (i) z_1^{10} (ii) z_2^{-4} (iii) $(z_1^*)^{10}$. Express the answers in the form $x + iy$.

7. Show that (i) $e^x = \cosh x + \sinh x$ (ii) $e^{-x} = \cosh x - \sinh x$

   (iii) $\cosh^2 x - \sinh^2 x = 1$ (iv) $\sin(iy) = i \sinh y$

   (v) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$.

8. Find (i) $2^{i/2}$ (ii) $\log_e i$ (iii) $(1+i)^{(1+i)}$.

   Care and patience is needed with part (iii). Start by writing $(1 + i)$ in exponential form. The answer to part (i) will come in useful.