Lines, Planes, Rotations – ANSWERS

1. (a) $\mathbf{n} = 5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$, $|\mathbf{n}| = \sqrt{50}$, so $\hat{\mathbf{n}} = \dfrac{5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}}{\sqrt{50}} = \dfrac{5}{\sqrt{50}}\mathbf{i} - \dfrac{4}{\sqrt{50}}\mathbf{j} - \dfrac{3}{\sqrt{50}}\mathbf{k}$.

   (b) When the equation of the plane is written in the form $\dfrac{5x - 4y - 3z}{\sqrt{50}} = \dfrac{10}{\sqrt{50}} = \sqrt{2}$, the rhs is the shortest distance to the plane from the origin; see Fact Sheet F.

   (c) Choose any point on the plane, say (2, 0, 0). The vector from (1, 3, 5) to (2, 0, 0) is $\mathbf{V} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$. The formula for the shortest distance is $s = |\mathbf{V}.\hat{\mathbf{n}}| = \dfrac{32}{\sqrt{50}}$.

2. We will refer to the normal **n** from the previous question as $\mathbf{n_1}$. A normal to the second plane is $\mathbf{n}_2 = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. The direction of the line of intersection is therefore $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{d} = -\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.
   A point that lies on both planes (and therefore on the line of intersection) is (–6, –10, 0). The vector equation of the line is therefore $\mathbf{r} = (-6\mathbf{i} - 10\mathbf{j}) + \lambda(-\mathbf{i} + \mathbf{j} - 3\mathbf{k})$. In terms of components, this is
   $\dfrac{x+6}{-1} = \dfrac{y+10}{1} = \dfrac{z}{-3} = \lambda$.

3. The equation $x - 2y - z = 14$ is a linear combination of the equations of the other two planes (the first plus twice the second). It will therefore share the same line of intersection.

4. The line has direction $\mathbf{d} = 7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and it follows that $\hat{\mathbf{d}} = \dfrac{7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}}{\sqrt{74}}$. The vector from (1, –2, 0) to e.g. (–2, 1, 2) is $\mathbf{V} = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. The shortest distance is $s = |\mathbf{V} \times \hat{\mathbf{d}}| = \left|\dfrac{\mathbf{i} + 23\mathbf{j} - 33\mathbf{k}}{\sqrt{74}}\right| = \sqrt{\dfrac{1619}{74}}$.

5. The vector from the reference point of the first line to the reference point of the second is $\mathbf{V} = (\alpha - 1)\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. The lines intersect when **V** and the two direction vectors are coplanar i.e. when $\begin{vmatrix} \alpha - 1 & -1 & -2 \\ 3 & 2 & 1 \\ 2 & -3 & -4 \end{vmatrix} = 0$. It follows that $\alpha = 17/5$.

6. (a) $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$ (b) $X = 7x - 4y, \quad Y = 2x.$

7. (a) $(2, \frac{1}{2})$ (b) (6, 3) (c) $(2, -1)$.

8. (a) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ (c) $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

9. (a) $\begin{pmatrix} -1/\sqrt{2} \\ 3/\sqrt{2} \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3/\sqrt{2} \\ 1/\sqrt{2} \\ 3 \end{pmatrix}$.

10. $\begin{pmatrix} 2\sqrt{2} \\ 1+\sqrt{2} \\ 1-\sqrt{2} \end{pmatrix}$