More Matrices

1. If $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & 2 & -1 \\ 4 & 1 & 3 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} -2 & 1 \\ 0 & 3 \\ 5 & 2 \end{pmatrix}$, find the following matrices where they exist:

   (a) $\mathbf{A}^{\mathrm{T}}$ (b) $\mathbf{A} + \mathbf{B}$ (c) $\mathbf{B} + \mathbf{C}^{\mathrm{T}}$ (d) $\mathbf{AB}$

   (e) $\mathbf{BC}$ (f) $(\mathbf{BC})^{\mathrm{T}}$ (g) $\mathbf{CB}$ (h) $\mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}$

2. If $\mathbf{A}$ is a matrix of order (shape) $r \times s$, what is the order of $\mathbf{AA}^{\mathrm{T}}$?

3. For any two matrices $\mathbf{B}$ and $\mathbf{C}$ for which the product $\mathbf{BC}$ can be defined, it can be shown that $(\mathbf{BC})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}$. Use this result to show that $\mathbf{AA}^{\mathrm{T}}$ is symmetric.

4. The three so-called "Pauli spin matrices" used in quantum theory are defined as

   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Find (i) $\sigma_1\sigma_1$ (ii) $\sigma_2\sigma_2$ (iii) $\sigma_3\sigma_3$ (iv) $\sigma_1\sigma_2 + \sigma_2\sigma_1$ (v) $\sigma_2\sigma_3 - \sigma_3\sigma_2$.

5. (Thanks are due to Dr Michael Coppins for supplying this highly edible question!)

   Boltzmann's Bakery uses the finest flour, margarine, sugar, currants, and eggs to make its celebrated range of products:

   Alternating Currant Cake (*a* made per day)
   Each cake uses 0.22 kg flour, 0.15 kg margarine, 0.15 kg sugar, 0.31 kg currants, and 2 eggs.
   Heaviside Layer Cake (*h* made per day)
   Each cake uses 0.22 kg flour, 0.20 kg margarine, 0.18 kg sugar, and 3 eggs.
   Fermat's Last Garibaldi Biscuits (*g* made per day)
   Each packet uses 0.20 kg flour, 0.08 kg margarine, 0.04 kg sugar, and 0.12 kg currants.
   XY Plain Biscuits (*p* made per day)
   Each packet uses 0.22 kg flour, and 0.05 kg margarine.

   The vector $\mathbf{p} = \begin{pmatrix} a \\ h \\ g \\ p \end{pmatrix}$ contains the numbers of each product made per day while $\mathbf{r} = \begin{pmatrix} r_f \\ r_m \\ r_s \\ r_c \\ r_e \end{pmatrix}$ represents the quantity of raw materials used per day (where f = flour, m = marg, s = sugar, c = currants, and e = eggs).

   Write down the matrix $\mathbf{B}$ such that $\mathbf{r} = \mathbf{Bp}$, and find $\mathbf{r}$ for $\mathbf{p} = \begin{pmatrix} 100 \\ 120 \\ 80 \\ 50 \end{pmatrix}$.