Fact Sheet J – Matrix Inversion

This fact sheet tells you how to find the inverse of a matrix, and takes you through an example in detail.

- Only non-singular square matrices have inverses, i.e. square matrices **A** for which $\det \mathbf{A} \neq 0$.
- When an inverse exists, the formula for the inverse is $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}\mathbf{A}}{\det \mathbf{A}}$ where adj **A** is the so-called *adjoint* of **A**.
- The *adjoint* of **A** is defined by $\operatorname{adj}\mathbf{A} = \mathbf{C}^{\mathrm{T}}$ where **C** is the *matrix of the cofactors* of **A**. The adjoint of **A** is therefore the transposed matrix of the cofactors.
- The matrix of the cofactors is obtained by replacing every element of **A** by its cofactor. The cofactor of element *jk* of **A** is the *minor* of that element multiplied by $(-1)^{j+k}$; the minor is the determinant of the matrix that remains when row *j* and column *k* of **A** are ignored.
- The following example is from question 2 of the Problems for Lecture 12:

The task is to find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix}$. The *determinant* of **A** is easily shown to be 2, which confirms that **A** has an inverse. The *matrix of the minors* is $\mathbf{M} = \begin{pmatrix} -7 & -2 & -3 \\ 8 & 2 & 4 \\ -11 & -2 & -5 \end{pmatrix}$ and, reversing the signs of the four "odd" elements (those for which *j*+*k* is odd) yields $\mathbf{C} = \begin{pmatrix} -7 & 2 & -3 \\ -8 & 2 & -4 \\ -11 & 2 & -5 \end{pmatrix}$ for the *matrix of the cofactors*. Hence the *adjoint matrix* is $\operatorname{adj}\mathbf{A} = \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -7 & -8 & -11 \\ 2 & 2 & 2 \\ -3 & -4 & -5 \end{pmatrix}$ and the *inverse matrix* is

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}\mathbf{A}}{\det \mathbf{A}} = \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix}.$$

It is readily verified that

$$\begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$