Fact Sheet H – Determinants (revised)

- A *determinant* (written $\det\{\mathbf{A}\}$ or $|A|$) is properly referred to as *the determinant of the matrix* $\mathbf{A}$. The determinant is defined only for a *square* matrix.
- The value of the 2×2 determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is $a_1 b_2 - a_2 b_1$.
- A 3×3 determinant is defined by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 \qquad (1)$$

Notice that every term contains an element from each row and an element from each column. All six possible combinations appear in the formula; the sign is positive for terms whose suffices are in cyclic order (1 2 3 1 2 3 ...), and negative otherwise.

The result can be written in several alternative (but entirely equivalent) forms e.g.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \qquad (2a)$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad (2b)$$

- In *double suffix notation*, eq.(2a) reads

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \qquad (3)$$

Notice that the first suffix denotes the row number and the second the column number.

- The 2×2 determinants in eqs.(2) and (3) are the so-called "minors" of the 3×3 parent determinant. In general, the *minor* M_{jk} of element *jk* of a determinant is the smaller determinant obtained when row *j* and column *k* of the original one are ignored.
- The *cofactor* C_{jk} is the so-called "signed minor" defined by $C_{jk} = (-1)^{j+k} M_{jk}$.
- In terms of minors and cofactors, eq.(3) can be written
- $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} M_{11} - a_{21} M_{21} + a_{31} M_{31} = a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} \qquad (4)$$

- A 4×4 determinant is defined by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} - a_{41}M_{41} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41} \quad (5)$$

  By this time, life is getting decidedly complicated!

- Note that, although the *value* of a determinant is unique, several different formulae can be used to work it out; for example eqs.(2a) and (2b) look different, but are in fact identical. And eq.(4) could equally be written

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \quad (6)$$

  It doesn't matter whether you use eq.(4) or eq.(6), the result will be the same. .

Properties of Determinants

The evaluation of determinants can be simplified by using the following general properties:

1. The sign of a determinant is reversed if two rows (or two columns) are interchanged
2. If all elements in any row (or column) are multiplied by a common factor f, the value of the determinant is multiplied by f.
3. The value of a determinant is zero if two rows (or two columns) are identical.
4. The value of a determinant is zero if two rows (or two columns) have proportional elements.
5. The value of a determinant is zero if any row (or any column) is made up exclusively of zeros.
6. The value of a determinant is unchanged if equal multiples of the elements of any row are added to the corresponding elements of any other row. The rule applies equally to columns.
7. The value of a determinant is unchanged if rows and columns are interchanged (i.e. if the underlying matrix is "transposed").
8. If the elements of any row (or column) are the sums of two (or more) terms, the determinant can be written as the sum of two (or more) determinants, for example,

$$\begin{vmatrix} a_1+u & b_1 & c_1 \\ a_2+v & b_2 & c_2 \\ a_3+w & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} u & b_1 & c_1 \\ v & b_2 & c_2 \\ w & b_3 & c_3 \end{vmatrix} \quad (7)$$

Most of the rules are readily deduced from the basic definitions of determinants. Rules 3 to 5 are logical consequences of rules 1 and 2. Rule 6 can be deduced from rule 8 if u, v, and w are proportional to elements of another column since, in that case, rule 4 implies that the second determinant on the right hand side of eq.(7) is zero.