Fact Sheet G
Linear Equations & Cramer's Rule

- Two *simultaneous linear equations* in two variables of the form

$$\left.\begin{aligned} a_1x + b_1y &= k_1 \\ a_2x + b_2y &= k_2 \end{aligned}\right\} \qquad (1)$$

have the solution

$$x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1} \qquad y = \frac{a_1k_2 - a_2k_1}{a_1b_2 - a_2b_1} \qquad (2)$$

- Using determinants, these solutions for x and y may be written

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad (3)$$

This result is known as Cramer's Rule.

- Note that the solutions for x and y have the same factor in the denominator, namely *the determinant of the coefficients* of the original equations.
- The definition of a 2×2 determinant is

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - rq \qquad (4)$$

Detailed information on determinants is given in Fact Sheet H.

- The solution given in eqs.(2) & (3) exists provided the factor in the denominator (the determinant of the coefficients) is non-zero. Each member of eqs.(1) is the equation of a straight line, and the solution is the point where the lines cross. However, if the lines are parallel, they never cross, and there is no solution; *this is the case where the determinant of the coefficients is zero*!
- For three simultaneous equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}\right\} \qquad (5)$$

the solutions are

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$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad (6)$$

This is Cramer's Rule for three equations. See Fact Sheet H for more information on 3×3 determinants.

- In matrix form, the sets of linear equations treated here read as follows

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \qquad (1a)$$

and

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \qquad (5a)$$

The 2×2 and 3×3 matrices on the left contain the coefficients of the original equations. The corresponding 2×2 and 3×3 determinants appear in the denominators of the solutions for *x*, *y*, and *z*.