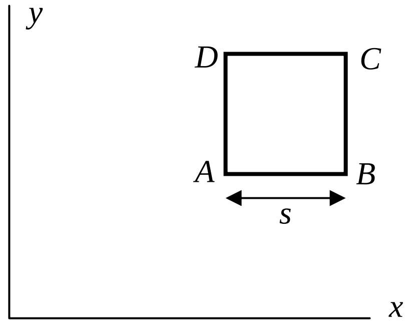
Classwork 7 *(updated)*
Transforming Areas … and Volumes

1. The matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms a point defined by the vector $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ into a new point defined by $\mathbf{R} = \begin{bmatrix} X \\ Y \end{bmatrix}$ through the equation $\mathbf{R} = \mathbf{M}\mathbf{r}$. Write down equations for X and Y in terms of x, y, and the elements of **M.** How is the origin transformed?

2. $ABCD$ is a square of side s as shown with the lower left-hand corner at $\mathbf{r}_{\mathrm{A}} = \begin{pmatrix} u \\ v \end{pmatrix}$.

   Write down expressions for (i) the vectors defining all four corners of the square and (ii) the vectors $\overrightarrow{AB}, \overrightarrow{DC}, \overrightarrow{AD}, \overrightarrow{BC}$.



3. Convince yourself that **M** transforms a straight line into another straight line. It follows that **M** transforms $ABCD$ into a quadrilateral $EFGH$. Write down the vectors defining all four corners of $EFGH$, and hence find the vectors $\overrightarrow{EF}, \overrightarrow{HG}, \overrightarrow{EH}, \overrightarrow{FG}$. You should find that $\overrightarrow{EF} = \overrightarrow{HG}$, which implies that opposite sides of the quadrilateral are equal in length and direction, and hence that the quadrilateral is a parallelogram.

4. If $\mathbf{r}_{\mathrm{A}} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $s = 3$, and $\mathbf{M} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$, find the corners of the parallelogram and the vectors $\overrightarrow{EF}, \overrightarrow{HG}, \overrightarrow{EH}, \overrightarrow{FG}$. Make a rough sketch of $ABCD$ and $EFGH$.

5. The area of a parallelogram is $|\mathbf{A} \times \mathbf{B}|$ where **A** and **B** are vectors defining two adjacent sides. If the parallelogram lies in the x-y plane, the area is therefore $|A_x B_y - A_y B_x|$. Use this result to find the area of $EFGH$, and show that the area scale factor for the transformation (i.e. the factor by which the area of the original square is increased) is $|ad - bc|$. Put in the numbers from the previous question.

6. Does the same area scale factor apply to other 2D shapes transformed by **M**?

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A further question for now or later …

7. These ideas can be extended to 3D. A unit cube has edges of unit length parallel to the coordinate axes and one corner is at the origin. The linear transformation $\mathbf{T} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ transforms the cube into a parallelepiped.

   (a) Write down the vectors representing the three edges of the parallelepiped that intersect at the origin.

   (b) By using the formula from Lecture 12 for the volume of a parallelepiped, find the volume scale factor for the transformation applied to the cube.

   (c) Does the same volume scale factor apply to other three-dimensional shapes transformed under this transformation?

Update

The original version quoted $s = 2$ in question 4. With this value, the answers are

$$\mathbf{r}_E = \begin{pmatrix} -5 \\ -6 \end{pmatrix}, \quad \mathbf{r}_F = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{r}_G = \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \quad \mathbf{r}_H = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

$$\overrightarrow{EF} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \overrightarrow{HG}, \quad \overrightarrow{EH} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \overrightarrow{FG}.$$