Classwork 4

Discover the “scalar triple product”

In this classwork, you will discover for yourself, in a few gentle steps, the so-called “scalar triple product” of three vectors, and you will explore its significance. The topic will be covered in a later lecture, but there’s no harm in finding out about it a bit ahead of time! Refer to the hints at the end if you have difficulty.

Note: In this Classwork, the symbol $|\;|$ refers to the magnitude of a vector, and has nothing to do with determinants. The magnitudes are there simply to ensure that the answers are positive.

At the start of Lecture 5, we saw that $|\mathbf{A}\times\mathbf{B}|$ represents the *area of a parallelogram* whose sides are **A** and **B**.

(a) (trivial) Consider a third vector **P,** which is parallel to $\mathbf{A}\times\mathbf{B}$. What can you say about the direction of **P** relative to the plane defined by **A** and **B**?

(b) What *geometrical* property does the scalar quantity $V_P = |\mathbf{P}||\mathbf{A}\times\mathbf{B}|$ represent?

(c) For a *general* vector **C**, what geometrical property does the scalar quantity $V = |\mathbf{C}.(\mathbf{A}\times\mathbf{B})|$ represent?

(d) Show that $V = |C_xA_yB_z - C_xA_zB_y + C_yA_zB_x - C_yA_xB_z + C_zA_xB_y - C_zA_yB_x|$

(e) Express V as a determinant.

(f) If $\mathbf{A} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{B} = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, and $\mathbf{C} = 2\mathbf{i} + s\mathbf{j} + t\mathbf{k}$, find V when $s = 1$ and $t = 2$.

(g) For what values of s and t is $V = 0$?

(h) For what values of s and t can **C** be written in the form $\mathbf{C} = \alpha\mathbf{A} + \beta\mathbf{B}$? For these values of s and t, find expressions for α and β.

(i) What is the geometrical significance of the conditions on s and t in (g) and (h)?

Hints

Part (b): The fact that the symbol used is V should give you a clue. Refer to part (a) for the significance of $\mathbf{A}\times\mathbf{B}$. Note that V_P is the product of two scalar quantities.

Part (c): This is a generalisation of the idea in part (b).

Part (d): Write out the three vectors in component form. Remember that the bars in the expression for V are there merely to ensure that V is a positive quantity, and have nothing to do with determinants.