Imperial College London

UNIVERSITY OF LONDON

MSc EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

MSP66 Lie Algebras and Classical Groups

Date: Friday 20th May 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Define a Lie Group.

   (b) Prove in detail that the circle S^1 can be given the structure of a Lie group.

2. (a) Let G be a Lie group. Define a left invariant vector field on G. Define the Lie algebra of G. Explain how the Lie algebra may be identified with the tangent space to the group at the identity element.

   (b) Let $U(n)$ be the group of unitary matrices. Find a matrix algebra isomorphic to the Lie algebra of $U(n)$. Hence find the dimension of $U(n)$.

3. (a) (i) Let G be a Lie group with Lie algebra $\mathcal{G}$. Define the exponential map $\exp : \mathcal{G} \to G$.

   (ii) Let $X \in \mathcal{G}$, and $s, t \in \mathbb{R}$. Show that $\exp((s+t)X) = \exp(sX)\exp(tX)$.

   (b) Let $M_n(\mathbb{R})$ be the Lie algebra of $n \times n$ matrices. Let $A \in M_n(\mathbb{R})$. Write down an expression for $\exp tA$. If there exists $X \in GL(n, \mathbb{R})$ such that XAX^{-1} is a diagonal matrix, show that $\det(\exp A) = e^{tr(A)}$.

4. (a) Let G be a Lie group contained in $GL(n, \mathbb{R})$, and let $g \in G$. Let X be a vector tangent to G at the identity, and let ϕ_{g*} be the derivative map. Find an expression for $\phi_{g*}(X)$.

   (b) Let $sl(2, \mathbb{R})$ be the Lie algebra of the group $SL(2, \mathbb{R})$. Write down a basis for $sl(2, \mathbb{R})$. Let
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$
   Calculate a matrix representing ϕ_{g*} with respect to your basis. Is the map $g \to \phi_{g*}$ for $g \in sl(2, \mathbb{R})$ one-one?

5. Let (x_1, x_2, x_3) and (y_1, y_2, y_3) denote elements of $\mathbb{R}^3$. Define a product
$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1\,,\ x_2 + y_2\,,\ x_3 + x_1 y_2 + y_3).$$

   (a) (i) Show that $\mathbb{R}^3$ together with the product above defines a Lie group G. (You need not check that G is a group.)

   (ii) Find the basis for the Lie algebra of G equal to the cartesian coordinate vector fields at the identity element. Write your basis with respect to the cartesian coordinate vector fields.

   (b) Show that the Lie algebra of G is not abelian but is nilpotent.