Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2006

MC1MF (Test)

Analytical Methods and Analysis

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (i) Define the function $y(x)$ to be

$$y(x) = \frac{2x^2 - 1}{x + 1} \ .$$

(a) Decompose this function into the sum of an even function and an odd function.

(b) Sketch a graph of $y(x)$ carefully indicating on your sketch any important features.

(ii) Find the derivative of the function $\sin(1 + x^2)$ from first principles.

(iii) (a) Find the value of the limit

$$\lim_{x\to\infty} \left(\frac{2\ \sinh\ x + \cosh\ x}{3\ \sinh\ x - \cosh\ x} \right) .$$

(b) Find the value of the limit

$$\lim_{x\to 0} (\cos x)^{1/x}.$$

Calculate the value of the definite integral

$$\int_0^1 \frac{e^x}{e^{2x} + 1}\, dx \ .$$

(iv) (a) Say whether the following statement is true or false:
If $A = \{x \in \mathbb{R} \mid x^3 < x^5\}$ and $B = \{x \in \mathbb{R} \mid 9 - x^2 > 0\}$ then
$A \cap B = \{x \in \mathbb{R} \mid -3 < x < 3\}$.

(b) For how many even integers n is $1.41\overline{42} - \sqrt{2n}$ rational?

(A) none (B) one (C) two (D) infinitely many

(c) Let $x = 2^{3/2}\, 5^{1/2}\, 4^{-1/4} \sqrt{10}$. Which of the following is true?

(A) $x < 10$ (B) $x^3 \in \mathbb{Q}$ (C) $x/\sqrt{5} \in \mathbb{N}$ (D) $x^2 > 10x + 2$

(d) How many complex numbers $z = x + iy$ with $x < 0$ satisfy $z^5 = 10$?

(A) 5 (B) 2 (C) 3 (D) 1 (E) infinitely many

SECTION B

2. Define the function $y(x)$ to be

$$y(x) = \sin^{-1}(x)$$

where it is specified that $y(0) = 0$.

(a) Find the first three non-zero terms in the Taylor series expansion of $y(x)$ about $x = 0$.

(b) Show that $y(x)$ satisfies the second order differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0 \ .$$

(c) By making use of the Leibniz rule, or otherwise, use the differential equation just derived to show that

$$y^{(n+2)}(0) = n^2 y^{(n)}(0)$$

where $y^{(k)}$ denotes the k-th derivative of $y(x)$ with respect to x.

(d) Hence show that the complete series expansion of $y(x)$ about $x = 0$ is

$$y(x) = \sum_{n=0}^{\infty} a_n x^{2n+1}$$

where

$$a_n = \frac{1}{(2n+1)!}\left(\frac{(2n)!}{2^n n!}\right)^2 .$$

(e) Verify that the values of a_0, a_1 and a_2 agree with the values you obtained in part (a).

3. (a) Define what it means for a real number to be irrational.

(b) Prove that if $a \neq 0$ is rational and b is irrational then ab is irrational.

(c) State the principle of induction.

(d) Prove that for every $n \in \mathbb{N}$, the integer $5^{2n} - 3^n$ is divisible by 11.

(e) Define the modulus $|z|$ of a complex number z.

(f) Let r be a rational number.
Prove that if $z = re^{2\pi i/11}$ and $m = 5^{200} - 3^{100}$, then $|z^m + r^m i|$ is irrational.
[You may assume that $\sqrt{2}$ is irrational.]