Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2008

This paper is also taken for the relevant examination for the Associateship.

MC1MF

Analytical Methods and Analysis

Date: May 2008 Time: ?

All questions carry equal weight.
Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

SECTION A

1. ## Part I

Select which of the five possible answers to each part is correct. You are not required to give any further explanation of your reasoning and any such reasoning will not be read.

(a) Let n be a positive integer and let ω be any nth root of unity. Which of the following statements are true?
$P_1 : \overline{\omega}^n$ is also an nth root of unity.
$P_2 : \overline{\omega} = \frac{1}{\omega}$.
$P_3 : \overline{(1-w)^n} = (w-1)^n$.
(i) none, (ii) all, (iii) P_1 only, (iv) P_1 and P_2 only, (v) P_3 only.

(b) Let r_1, r_2 and r_3 be the following real numbers.
$r_1 = 0.1b_2b_3\dots b_n\dots$ where $b_i = 1$ if i is prime and $b_i = 0$ otherwise.
$r_2 = (\sqrt{2} - \sqrt{8})^2$.
$r_3 = (\sqrt{2} - \sqrt{3})^2$.
Which of these 3 real numbers are irrational?
(i) none, (ii) all, (iii) r_1 and r_3 only, (iv) r_1 only, (v) r_2 and r_3 only.

(c) Let x, y and z be nonzero real numbers. Which of the following three statements are true?
$P_1 : x > y \Rightarrow xz > yz$.
$P_2 : x > y \Rightarrow x^2 > y^2$.
P_3 : if $x + y + z = 0$ then $xy + yz + xz < 0$.
(i) none, (ii) all, (iii) P_1 and P_3 only, (iv) P_3 only, (v) P_2 only.

Part II

The function $f(x)$ is defined for positive values of x by

$$f(x) = x^{1/x}.$$

(d) Write $f(x) = \exp[g(x)]$ for a suitable function $g(x)$.

(e) Find the limits of $f(x)$ as $x \to 0$ and as $x \to \infty$.

(f) Find the maximum value attained by $f(x)$ for $x > 0$.

(g) Sketch the curve $y = f(x)$ for $x > 0$.

(h) Determine whether or not the integral

$$\int_0^\infty \frac{(x+1)\sin x}{x^{3/2}(x-\pi)}\,dx$$

exists (do NOT try to evaluate it.)

(i) Solve the differential equation

$$e^x \frac{dy}{dx} = \sinh x \cos^2 y.$$

SECTION B

2. (a) Give precise statements of the Principle of Mathematical Induction and the Principle of Strong Mathematical Induction.

(b) For any integer $n \geq 3$ prove that the sum of the internal angles in an n-sided polygon is $(n-2)\pi$. (You may assume that the sum of the internal angles of any triangle is π.)

(c) Prove that between any two distinct rational numbers there is an irrational number and also another rational number. (You may assume that $\sqrt{2}$ is irrational, but you should prove all other properties of rational and irrational numbers that you use).

(d) Find all real or complex solutions of the simultaneous equations

$$\begin{aligned} x + y - z &= 3 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 - z^3 &= 3. \end{aligned}$$

(Try looking for a cubic equation with roots closely related to x, y and z.)

3. For some positive integer n, x and y are defined in terms of a parameter θ by

$$x = \cos\theta, \qquad y = \cos(n\theta).$$

(a) Using De Moivre's Theorem and the Binomial Theorem, show that

$$y = \sum_{m=0}^{p} \binom{n}{2m} x^{n-2m}(x^2-1)^m,$$

where p is the largest integer with $2p \leqslant n$ and $\binom{k}{l}$ denotes a binomial coefficient.

(b) Show that

$$\sin\theta\, y' = n\sin n\theta$$

where $y' = dy/dx$ and hence that $y(x)$ obeys

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

Using Leibniz' formula, deduce that

$$y^{(k+2)}(0) = (k^2 - n^2)y^{(k)}(0) \qquad \text{for} \quad k \geqslant 0,$$

where $y^{(k)}(0)$ denotes the k'th derivative of y with respect to x evaluated at $x = 0$.

If n is a multiple of 4 show that $y(0) = 1$ and $y'(0) = 0$. Hence find the Maclaurin series for the function $y(x)$ when n is a multiple of 4, including terms up to x^4.

Verify that parts (a) and (b) give the same answer when $n = 4$.

4. (a) Rolle's theorem states that if $h(x)$ is a continuous function, is differentiable in $a < x < b$ and if $h(a) = h(b)$, then there exists a value ξ such that $a < \xi < b$ and $h'(\xi) = 0$.

Given two differentiable functions $f(x)$ and $g(x)$, consider the function

$$h(x) = f(x) + mg(x)$$

where m is a suitably chosen constant. Use Rolle's theorem to prove that provided $g(a) \neq g(b)$ there exists a ξ in $a < \xi < b$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Deduce that if $f(c) = 0 = g(c)$, then

$$\lim_{x \to c} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \to c} \left[\frac{f'(x)}{g'(x)} \right],$$

assuming both limits exist.

(b) Using any method, find the limit

$$\lim_{x \to 0} \left[\frac{1 - \cos x - \sin x + \log(1 + x)}{(1 + x^3)^{1/2} - 1} \right].$$

(c) The functions $f(x)$, $f_e(x)$ and $f_o(x)$ are defined by

$$f(x) = \frac{1}{1 - x} = f_e(x) + f_o(x),$$

where f_e is an even function and f_o is an odd function. Sketch the three functions f, f_e and f_o.