1. (a) Define what it means for a real number to be rational and for a real number to be irrational.

(b) Prove that if a is irrational and b is a non-zero rational then ab is irrational.

(c) Prove that between any 2 distinct rational numbers there is an irrational number.

(d) For each of the following 3 statements either prove that it is true or prove that it is false.

(i) $\sqrt{2} + \sqrt{11} < 5$.
(ii) $\sqrt{60} + \sqrt{5/2}$ is irrational.
(iii) Let x be the real number whose decimal expansion is

$$x = 0.a_1 a_2 a_3 \ldots$$

where $a_n = 7$ if n is divisible by 7 and $a_n = 0$ otherwise. Then x is irrational.

(e) Find a quadratic equation whose roots are $7+2\sqrt{10}$ and $7-2\sqrt{10}$. Hence, or otherwise, find a quartic equation with integer coefficients satisfied by $\sqrt{5}+\sqrt{2}$ and $\sqrt{5}-\sqrt{2}$.

2. (a) Use De Moivre's theorem to prove

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2,$$

and to prove the analogous formula for $\sin(\theta_1 + \theta_2)$.

(b) Use part (a) to find a formula for $\tan(\theta_1 + \theta_2)$ in terms of $\tan\theta_1$ and $\tan\theta_2$.

(c) Using the previous part, or otherwise, find an expression for $\tan(\pi/8)$, and use this to prove that $\tan(\pi/8)$ is irrational.

(d) Show that if p is a polynomial with real coefficients and α is a root of p, then so is $\bar{\alpha}$. Use this fact to prove that every real polynomial factorizes as a product of real linear and real quadratic polynomials. (You may use the fact that every complex polynomial of degree n factorizes as a product of linear factors and has exactly n roots, counting repeated roots).

(e) Find all solutions to the equation

$$|z - a| = |z - b|,$$

where a and b are distinct complex numbers. Give a geometric interpretation of these solutions. (Hint: you might want to check that the midpoint of the line segment joining a and b is a solution of this equation.)

3. (a) State precisely what it means for a function $f(x)$ to be differentiable at $x = a$. If $f(x)$ and $g(x)$ are differentiable at $x = a$, prove **from first principles** that $(fg)' = f'g + fg'$.

(b) Evaluate the limit:

$$\lim_{x \to 0} \frac{\log(1 + \sin x) - x}{\exp(x^2) - \cos x}.$$

(c) Find an even function $f(x)$ and an odd function $g(x)$ such that

$$f(x) + g(x) = 2^x.$$

Show that $f'(1) = (\log 2) g(1)$ and hence or otherwise show that

$$\lim_{x \to 1} \left(\frac{g(x-1)}{f(x) - f(1)} \right) = \frac{4}{3}.$$

4. (a) Find a formula for the n^{th} derivative of the function $f(z) = \log(z + \frac{1}{2})$, for $n \geqslant 1$. Hence derive a series expansion of $f(z)$ about $z = 0$, including the general term. State the ratio test carefully, and derive the radius of convergence of this series.

(b) If $f(z)$ is as above and x is a real number, show that

$$\Re e \left[f(\tfrac{1}{2} e^{2ix}) \right] = \log |\cos x|$$

where $\Re e$ denotes the real part, and find the corresponding imaginary part.

(c) Assuming the series expansion of part (a) is valid when $z = \frac{1}{2} e^{2ix}$, obtain the relation

$$x = \sin 2x - \tfrac{1}{2} \sin 4x + \tfrac{1}{3} \sin 6x + \ldots (-1)^{(n-1)} \frac{\sin 2nx}{n} + \ldots$$

Comment on the behaviour of this expression when $x = 0$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$ and $\frac{3}{4}\pi$.

5. (a) The shape of a long icicle hanging from the ceiling is given by the function $R(z)$, where R is the radius of the circular cross-section at height z above the ground. It can be shown that in terms of an angle ψ, that

$$\frac{dR}{dz} = \tan\psi \qquad \text{and} \qquad z = \frac{1}{\sin^4\psi}$$

Given that $R = 0$ at $z = 1$, show that

$$R = \tfrac{4}{3}(\sqrt{z} - 1)^{1/2}(2 + \sqrt{z}).$$

(b) Find $y(x)$ satisfying

$$\frac{dy}{dx} + e^x y = e^x \qquad \text{with} \qquad y(0) = 0.$$

(c) Find $y(x)$ satisfying

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right).$$