UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

MC1MF

Analytical Methods and Analysis

Date: Wednesday, 10th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) The function $f(x)$ is defined as

$$f(x) = \cosh(x + x^2)\,.$$

(i) Write $f(x)$ as the sum of an even function and an odd function.

(ii) Find the first three non-zero terms in the series expansion of $f(x)$ about $x = 0$.

(iii) Find the derivative of $f(x)$ from first principles.

(b) Sketch the curve defined by the relation

$$y^2 = x^3(1 - x^3),$$

carefully indicating any important features on your sketch.

2. Define

$$f(x) = \tanh^{-1}(x)\,.$$

(a) Find an expression for $f(x)$ in terms of the logarithm function.

(b) Hence, or otherwise, show that the n-th derivative of $f(x)$, for $n \geq 1$, is given by

$$\frac{d^n f}{dx^n} = \frac{(n-1)!}{2}\left(\frac{(-1)^{n-1}}{(1+x)^n} + \frac{1}{(1-x)^n}\right)\,.$$

(c) Find the complete Taylor series of $f(x)$ about $x = 0$.

(d) Let the function $F(x)$ be defined by

$$F(x) = \int_0^x f(x)\ dx\,.$$

By using integration by parts to find $F(x)$ explicitly, show that

$$F(1/2) = \log\left(\frac{3^{3/4}}{2}\right)\,.$$

3. (a) Find the general solution of the equation

$$\frac{d^2T}{dr^2} - \frac{2}{r}\frac{dT}{dr} = r^2.$$

(b) Find the general solution of the equation

$$\left(\frac{x+y}{x-y}\right)\frac{dy}{dx} = 1.$$

(c) Find the general solution of the equation

$$\frac{dy}{dx} = \sec x\ \sec y\,.$$

4. (a) Define what it means for a real number to be *irrational*.

   (b) Prove that $\sqrt{2}$ is irrational.

   (c) Deduce that if a, b, c, d are integers such that $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$.

   (d) If α and β are rational, and γ and δ are real, then does $\alpha + \beta\sqrt{2} = \gamma + \delta\sqrt{2}$ imply $\alpha = \gamma$ and $\beta = \delta$. (Proof or counterexample required.)

   (e) For each $n \in \mathbb{N}$, the integers a_n and b_n are defined by the formula $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$. Prove by induction that a_n is odd for all n, and that b_n is even if and only if n is even.

5. (a) Define the *modulus* $|z|$ of a complex number z.

   (b) Draw a clear sketch of each of the following sets of complex numbers $A = \{z \,:\, |z| = |z + 2i|\}$ and $B = \{z \,:\, |z + i| = \sqrt{3}\}$.

   (c) Prove that if $\omega \in A \cap B$ then $\omega^6 = -64$.

   (d) By taking real parts in Moivre's Theorem, find a formula for $\cos 4\theta$ in terms of $\cos\theta$.

   (e) Use your formula from (d) to find a quartic (i.e. degree 4) equation with real coefficients that has $\cos(\pi/12)$ as a root.