1.(a) Define what it means for a real number to be irrational.

(b) Give an example of two real numbers α and β such that $\alpha+\beta$ is rational but $\alpha\beta$ is irrational, stating clearly any facts that you use.

(c) State the principle of mathematical induction.

(d) Prove that $5^{2n} - 3^n$ is divisible by 11 for all integers $n \geq 1$.

(e) Let x be the real number whose decimal expansion is

$$x = 0 \cdot a_1 a_2 a_3 \ldots$$

where $a_n = 2$ if n^2 is divisible by 9 and $a_n = 0$ otherwise. Prove that x is irrational or express it as a fraction.

2. (a) Define the *modulus* $|z|$ and *complex conjugate* $\overline{z}$ of a complex number z.

(b) Prove that if $|z| = 1$ then $\overline{z} = \frac{1}{z}$.

(c) Let t be a complex number and let n be a positive integer. Stating any facts that you use clearly, explain briefly why there are exactly n complex numbers z such that $z^n = t$ and describe them.

(d) Prove that if $\omega^6 = 1$ and $\omega^2 \neq 1$ then $\omega + \frac{1}{\omega} \in \{-1, 1\}$.

3. Define the function

$$f(x) = \frac{1}{x}\left(\sqrt{x^2+1} - 1\right).$$

where the positive square root is assumed.

(a) Find the first three non-zero terms in the Taylor series expansion of this function about $x = 0$;

(b) Show that, as $x \to +\infty$,

$$f(x) \to 1 - \frac{1}{x} + \frac{1}{2x^2} + \ldots$$

(c) Does $f(x)$ have any stationary points in the domain $x > 0$?

(d) Sketch a graph of $f(x)$ for $x > 0$.

4. (a) Find the following indefinite integrals:

$$(i) \int x \tan^{-1} x \; dx;$$

$$(ii) \int \frac{e^x + 1}{e^x - 1} dx.$$

(b) Define

$$I_n = \int_0^{\pi/2} \sin^{2n} x \; dx.$$

Show that

$$I_n = \left(\frac{2n-1}{2n} \right) I_{n-1} \quad \text{for } n \geq 1.$$

Hence, show that

$$I_n = \frac{\pi (2n)!}{2^{2n+1} (n!)^2}.$$

5. (a) Find the general solution of the equation

$$x\frac{dy}{dx} = 1 + y^2.$$

(b) Find the solution of

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = x^3$$

satisfying the conditions $y(1) = 1, y'(1) = 0$.

(c) Find the general solution of

$$\frac{dy}{dx} = \frac{1}{x + y^2}.$$