UNIVERSITY OF LONDON

IMPERIAL COLLEGE LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2007

This paper is also taken for the relevant examination for the Associateship.

M4P46 MSP66 Lie algebras

DATE: examdate TIME: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Setter's signature ....

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In this paper all Lie algebras are defined over the field of complex numbers. Let n be a positive integer.

1. *i)* Give the definition of a nilpotent Lie algebra.

ii) Prove that every subalgebra of a nilpotent Lie algebra is nilpotent.

iii) Prove that the set of $n \times n$-matrices $A = (a_{ij})$ such that $a_{ij} = 0$ for $i \geq j$, is a nilpotent Lie algebra.

iv) What can you say about a nilpotent Lie algebra whose centre is zero?

2. *i)* Give the definition of the adjoint representation and the Killing form of a Lie algebra.

ii) Working from the definition of the Killing form find the matrix of the Killing form of $\mathfrak{sl}(2)$ in the standard basis H, X_+, X_-.

iii) Using the result of (iii), or otherwise, prove that $\mathfrak{sl}(2)$ is a semisimple Lie algebra. (You can use any results from the course provided you state them clearly.)

3. *i)* Give the definition of a root system and of the associated Weyl group.

ii) Prove that the Weyl group is finite.

iii) Give the definition of the root system D_5. What is the number of roots in this root system? What is the dimension of a semisimple Lie algebra of type D_5?

iv) Find the order of the Weyl group of the root system D_5. (Justify your answer.)

4. *i)* Give the definition of a Cartan subalgebra of a Lie algebra.

ii) Give an example of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(4)$. (No proof is required.)

iii) Decompose $\mathfrak{sl}(4)$ into a direct sum of vector spaces $\mathfrak{g}_\alpha$, where α are linear functions $\mathfrak{h} \to \mathbb{C}$, such that $\mathrm{ad}(x)v = \alpha(x)v$ for any $x \in \mathfrak{h}$ and $v \in \mathfrak{g}_\alpha$. (Justify your answer.)

iv) Using (iii) identify the root system of $\mathfrak{sl}(4)$. (Justify your answer.)

5. *i)* List the Cartan matrices of all inequivalent irreducible root systems of rank 4, and identify the corresponding root systems. (No proof is required.)

ii) Prove that in an irreducible root system the lengths of the roots are either all equal, or take two different values.

You may use any results from the course as long as you clearly state them.