1. Define the group $\text{Aff}(\mathbf{R}^2)$ of affine transformations of $\mathbf{R}^2$. Show that there are group homomorphisms

$$\alpha : \mathbf{R}^2 \to \text{Aff}(\mathbf{R}^2)\ ,\ \beta : \text{Aff}(\mathbf{R}^2) \to GL(2, \mathbf{R})$$

such that α is injective and β is surjective.

Define the real projective space $\mathbf{RP}^2$ and the group $PGL(3, \mathbf{R})$ of projective transformations of $\mathbf{RP}^2$. Give the definition of a *projective line* in $\mathbf{RP}^2$. Show that if L is a projective line and g is an element of $PGL(3, \mathbf{R})$ then $g(L)$ is also a projective line. Show that, for any line L, the subgroup $\{g \in PGL(3, \mathbf{R}) : g(L) = L\}$ is isomorphic to $\text{Aff}(\mathbf{R}^2)$.

Suppose that $\Gamma \subset \text{Aff}(\mathbf{R}^2)$ is a *finite* subgroup, of order d. By considering the action of Γ on the point $d^{-1}\left(\sum_{\gamma\in\Gamma} \gamma(0)\right)$, show that Γ is contained in a subgroup G of $\text{Aff}(\mathbf{R}^2)$ such that the restriction of β gives an isomorphism from G to $GL(2, \mathbf{R})$.

2. Let U be an open subset of $\mathbf{R}^2$ and E, F, G be smooth functions on U with $E > 0, G > 0, EG - F^2 > 0$. Explain how the Riemannian metric

$$g = Edx^2 + 2Fdxdy + Gdy^2$$

defines the length $L_g(\gamma)$ of a path γ in U and the distance $d_g(p, q)$ between two points p, q in U.

Now make the standard identification of $\mathbf{R}^2$ with $\mathbf{C}$ and let H be the upper half-plane $H = \{x + iy : y > 0\}$. Let g be the Riemannian metric

$$g = \frac{1}{y^2}\left(dx^2 + dy^2\right)$$

on H. Show that if λ, μ are real numbers with $\lambda > \mu > 0$ then

$$d_g(\lambda i, \mu i) = \log\lambda - \log\mu.$$

Let a, b, c, d be real numbers with $ad - bc > 0$ and let f be the Mobius map

$$f(z) = \frac{az + b}{cz + d}.$$

Show that $d_g(z, w) = d_g(f(z), f(w))$, for any two points $z, w \in H$.

By considering a suitable Mobius map, show that for any real numbers θ, μ with $\mu > 0$

$$d_g\left(i, \frac{\sin\theta + i\mu\cos\theta}{\cos\theta - i\mu\sin\theta}\right) = |\log\mu|.$$

3. Give the definitions of a *smooth manifold* and a *Lie group*. Show that the real projective plane $\mathbf{RP}^2$ is a smooth manifold. Let $Q \subset \mathbf{RP}^2$ be a non-empty, non-singular conic. Show that $\mathbf{RP}^2 \setminus Q$ is the disjoint union of connected components Ω, Ω^*, where Ω is homeomorphic to a disc and there is a surjective, two-to-one map $\pi : S^1 \times \mathbf{R} \to \Omega^*$. Find a Lie group which acts on $\mathbf{RP}^2$ with three distinct orbits Q, Ω, Ω^*.

4. Let $M_n(\mathbf{R})$ denote the set of $n \times n$ matrices with real entries. Define the exponential $\exp(A)$ of a matrix $A \in M_n(\mathbf{R})$. [You may assume that $\|AB\| \leq \|A\| \, \|B\|$, where $\|A\|^2 = \sum_{ij} A_{ij}^2$.]
Show that

$$\frac{d}{dt} \exp(tA) = A \exp(tA)$$

and deduce that

$$\det(\exp(A)) = e^{\mathrm{Tr}(A)}.$$

Now suppose that $G \subset GL(n, \mathbf{R})$ is a a subgroup and also a submanifold, with tangent space $TG_1 \subset M_n(\mathbf{R})$ at the identity. Show that for $A, B \in TG_1$ the bracket $AB - BA$ is also in TG_1. [You may assume without proof that for any $A \in TG_1$ the exponential $\exp(A)$ lies in G.]

5. Give the definition of a *connection* (or *covariant derivative*) ∇ on the tangent bundle of a smooth manifold M. Define what it means for ∇ to be *torsion-free*, and for ∇ to be compatible with a Riemannian metric on M.

Let H be a Lie group. Explain how an element of H acts by left-translation on tangent vectors to H. Let $\langle \ , \ \rangle$ be a Euclidean inner product on TH_1 and let g be the left-invariant Riemannian metric on H, equal to $\langle \ , \ \rangle$ on TH_1. Explain why there is a bilinear map

$$B : TH_1 \times TH_1 \to TH_1$$

characterized by the condition that

$$\langle B(x,y), z \rangle = \langle [z,x], y \rangle + \langle x, [z,y] \rangle$$

for all $x, y, z \in TH_1$. Show that B is zero if the metric g is also invariant under right translation. If ∇ is the torsion-free connection on TH which is compatible with g, show that

$$2\nabla_X Y = [X,Y] + B(X,Y),$$

for left-invariant vector fields X, Y.