UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

M4P43/MSP13

Algebraic Topology

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UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

M4P43/MSP13

Algebraic Topology

Date: Friday, 26th May 2007 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Let X be a topological space and A be a subspace, $A \subset X$. Give the definition for A to be

   (a) A retract of X.

   (b) A deformation retract of X.

   (ii) Give an example of X and a connected subset $A \subset X$ such that

   (a) A is not a retract of X.

   (b) A is a retract of X but A is not a deformation retract of X.

   Give brief justifications.

2. (i) (a) Let X and Y be two path-connected topological spaces. Express the fundamental group of the space $X \times Y$ in terms of the fundamental groups of X and Y.

   (b) Explain with the help of an appropriate picture why the fundamental group of the complement of two points in $\mathbb{R}^2$ is $\mathbb{Z} * \mathbb{Z}$, where $*$ denotes the free product.

   (ii) Consider in $\mathbb{R}^3$ two vertical lines L_1, L_2 and a horizontal circle S^1 given by $L_1 = \{(0,0,t)|t \in \mathbb{R}\}$, $L_2 = \{(0,1,t)|t \in \mathbb{R}\}$, and $S^1 = \{(\cos \mathrm{t}, \sin \mathrm{t}, 0)|\mathrm{t} \in [0, 2\pi]\}$ respectively. Find the fundamental groups of the following complements in $\mathbb{R}^3$.

   (a) $\mathbb{R}^3 \setminus \{L_1 \cup L_2\}$.

   (b) $\mathbb{R}^3 \setminus \{L_1 \cup S^1\}$.

3. (a) Let X be a path-connected topological space and let D^2 be a disk with the boundary S^1. For a continuous map $\phi : S^1 \to X$ let Y be the space obtained from X by attaching to it D^2 along the map ϕ. Let x be a point in S^1. State the relation between the fundamental groups $\pi_1(X, \phi(x))$ and $\pi_1(Y, \phi(x))$. (No proof is needed here).

   (b) Show how to give the Klein bottle the structure of a cell complex.

   (c) Using part (a) or otherwise give a presentation of the fundamental group of the Klein bottle by generators and relations .

4. (i) Let X be a path-connected space and let $p : \widetilde{X} \to X$ be a covering map. Let x be a marked point on X.

   (a) Show that any loop $\gamma : [0,1] \to X$ based at x gives an action on $p^{-1}(x)$. Explain without proof how this action gives rise to the action of $\pi_1(X,x)$ on the set $p^{-1}(x)$.

   (b) Prove that the space $\widetilde{X}$ is path-connected iff the action of $\pi_1(X,x)$ on $p^{-1}(x)$ is transitive.

   (ii) Find the number of non-isomorphic connected degree 3 covers of the 2-torus T^2. Explain your reasoning.

5. (i) Consider the n-simplex Δ^n, $\Delta^n = [v_0, ..., v_n]$.

   (a) Give the formula for the boundary $\partial_n(\Delta^n)$.

   (b) Prove that $\partial_{n-1}\partial_n(\Delta^n) = 0$.

   (ii) Let X be the Δ-complex obtained from two 2-simplexes $[v_0, v_1, v_2]$ and $[u_0, u_1, u_2]$ by making the following identifications of edges:

$$[v_0, v_1] \simeq [u_0, u_1],$$

$$[v_0, v_2] \simeq [v_1, v_2] \simeq [u_0, u_2] \simeq [u_1, u_2]$$

Compute the simplicial homology groups of X.