Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MSc EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M4P43/MSP13

Algebraic Topology

Date: Friday, 26th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Let X be a topological space and let $f_0, f_1 : I \to X$ be two paths with same end points. What does it mean for f_0 and f_1 to be *homotopic*?

(ii) For any two paths in $\mathbb{R}^n$ with same endpoints give a formula for a homotopy between them.

(iii) State the Brouwer fixed point theorem in dimension 2.

(iv) Consider two loops in $\mathbb{R}^2 \setminus \{(0,0)\}$, $f, g : [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$, given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$, $g(t) = (2 - \cos(2\pi t), \sin(2\pi t))$. Prove that these loops are not homotopic in $\mathbb{R}^2 \setminus \{(0,0)\}$.

2. (i) (a) Let X be a path connected topological space and $Y \subset X$ be a deformation retract of X. What can you say about groups $\pi_1(X)$ and $\pi_1(Y)$? (No proof is needed here).

(b) Let X and Y be two path connected topological spaces. What is the relation between the groups $\pi_1(X)$, $\pi_1(Y)$ and $\pi_1(X \vee Y)$? (No proof is needed here.)

(ii) Consider a disk D^2 and let x_1, x_2 be two points in the interior of D^2.

(a) Prove that there is a subset in $D^2 \setminus \{x_1, x_2\}$ homeomorphic to $S^1 \vee S^1$ and such that $D^2 \setminus \{x_1, x_2\}$ deformation retracts on it. (In this question you can give a proof by giving an appropriate picture.)

(b) Prove that ∂D^2 is not a deformation retract of $D^2 \setminus \{x_1, x_2\}$.

3. (i) Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map.

(a) Define what is means for a map $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ to be a *lift* of f?

(b) Suppose that Y is connected and locally path connected. What condition on the subgroups $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ and $f_*(\pi_1(Y, y_0))$ of $\pi_1(X, x_0)$ must be satisfied for f to have a lift? (No proof is needed here.)

(ii) Let $f : S^2 \to T^2$ be a continuous map.

(a) Prove that the map f has a lift to a map from S^2 to the universal cover of T^2.

(b) Deduce that f is homotopic to a constant map from S^2 to T^2.

4. (i) Consider the following sequence of homomorphisms of abelian groups:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_n} C_n \xrightarrow{\partial_{n-1}} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

(a) Give the definition for it to be a *chain complex*.

(b) Give the definition for it to be an *exact sequence*.

(ii) Compute the simplicial homology groups of the Δ-complex obtained from a 3-dimensional simplex $\Delta^3 = [v_0, v_1, v_2, v_3]$ by identifying all 6 edges of Δ^3 to a single edge.

5. Let X be a cell complex with a finite number of cells.

(a) Give the definition of the *Euler characteristic* $\chi(X)$ of X.

(b) Let $p : \widetilde{X} \to X$ be a n-sheet covering. Prove that $\chi(\widetilde{X}) = n\chi(X)$.

(c) Suppose that there is a covering map $p : S^{2n} \to X$. Prove that the possible number of sheets for such covering may be only 1 or 2. Give examples of X and p for each case.