Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M4P43/MSP13 Algebraic Topology

Date: Thursday, 19th May 2005 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let

$$Y = S^1 \vee S^1 = \{z_1, z_2 \in S^1 \times S^1 \mid z_1 = 1 \text{ or } z_2 = 1\}$$

with basepoint $y = (1,1) \in Y$. (We consider S^1 as the set of complex numbers z with $|z| = 1$).

Let

$$X = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1 \in \mathbb{Z} \text{ or } a_2 \in \mathbb{Z}\}$$

with basepoint $x = (0,0)$.

Let $f : X \to Y$ be the map defined by

$$f(a_1, a_2) = (e^{2\pi i a_1}, e^{2\pi i a_2})$$

(a) What does it mean for a map $g : V \to W$ of topological spaces to be a covering map?

Prove that $f : X \to Y$ is a covering map.

(b) Calculate the corresponding action of $\pi_1(Y, y)$ on $f^{-1}(y)$. (You may assume that $\pi_1(Y, y)$ is the free group on two generators).

(c) Let $a, b \in \pi_1(Y, y)$ denote the generators of $\pi_1(Y, y)$ corresponding to the two circles in Y. Show that the subgroup $f_*(\pi_1(X, x)) \subset \pi_1(Y, y)$ is the set

$$\{a^{m_1} b^{n_1} \dots a^{m_k} b^{n_k} \mid m_i, n_i \in \mathbb{Z},\ \sum m_i = 0, \sum n_j = 0\}$$

2. (a) State Van Kampen's theorem.

(b) Compute the fundamental group of Σ_g, the compact oriented surface of genus g. (You may assume any properties you like about the fundamental group of S^1 and of the wedge product $S^1 \vee \dots \vee S^1$ of n circles, as long as they are stated correctly).

3. (a) What is the n simplex $\triangle_n$?

Let X be a topological space. Define the group $C_n(X)$ of singular n chains on X and the boundary map $\mathrm{d} : C_n(X) \to C_{n-1}(X)$.

Let $f : X \to Y$ be a map of spaces. Define the induced map $f_* : C_n(X) \to C_n(Y)$. How do the maps f_* and the boundary maps d on $C_*(X)$ and $C_*(Y)$ interact?

(b) Let X, Y be topological spaces, and let $f, g : X \to Y$ be homotopic maps.

Construct the maps $S : C_n(X) \to C_{n+1}(Y)$ which give a chain homotopy between f_* and g_*. Briefly sketch the proof that S satisfies the defining equation of a chain homotopy,

$$\mathrm{d}\, S + S\,\mathrm{d} = f_* - g_*$$

4. (a) What is the Mayer-Vietoris sequence? (You need not define the maps in the sequence).

Use the Mayer-Vietoris sequence to calculate the homology groups of S^1.

(b) Let X be a topological space. Use the Mayer-Vietoris sequence to show that the homology groups of $X \times S^1$ are

$$H_i(X \times S^1) = H_i(X) \oplus H_{i-1}(X)$$

for all $i \in \mathbb{Z}$.

(You may assume any basic properties of homology).

5. (a) Define the group $C_k^{cell}(X)$ of cellular chains of a cell complex X, and the differential

$$\mathrm{d} : C_k^{cell}(X) \to C_{k-1}^{cell}(X)$$

(b) Define the complex projective space $\mathbb{CP}^n$.

Give $\mathbb{CP}^n$ the structure of a cell complex and hence compute its homology groups.