Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MSc EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M4P42/MSP12 Analysis on Manifolds and Heat Kernels

Date: Thursday, 1st June 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the following differential operator in Ω

$$\mathcal{L} = \Delta + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x^j},$$

where $b_j(x)$ are bounded functions in Ω and $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{(\partial x^j)^2}$ is the Laplace operator.

(a) Prove that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $\mathcal{L}u > 0$ in Ω then

$$\sup_{\Omega} u = \sup_{\partial\Omega} u. \tag{1}$$

(b) Show that there exists a function $v \in C^2(\mathbb{R}^n)$ such that $\mathcal{L}v > 0$ in Ω.

(c) Prove that (1) holds also for any function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\mathcal{L}u \geq 0$ in Ω.

(d) Prove that, for any function f on Ω and any function g on $\partial\Omega$, there is at most one function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solving the boundary value problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

2. Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and $p = (0, 0, \ldots, -1)$ be the south pole of the sphere. The stereographic projection is the mapping P from $\mathbb{S}^n \setminus \{p\}$ to the subspace

$$\mathbb{R}^n = \left\{x \in \mathbb{R}^{n+1} : x^{n+1} = 0\right\},$$

which is defined as follows: if $x \in \mathbb{S}^n \setminus \{p\}$ then Px is the point of the intersection of $\mathbb{R}^n$ with the straight line through p and x.

(a) Prove that $Px = \frac{x'}{x^{n+1}+1}$ for any $x = (x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \setminus \{p\}$, where $x' = (x^1, \ldots, x^n)$. Show that P is a bijection of $\mathbb{S}^n \setminus \{p\}$ onto $\mathbb{R}^n$.

(b) Consider the Cartesian coordinates $y^1, \ldots, y^n$ in $\mathbb{R}^n$ as local coordinates on $\mathbb{S}^n \setminus \{p\}$ using the pullback by the stereographic projection. Prove that the canonical spherical metric $\mathbf{g}_{\mathbb{S}^n}$ has in these coordinates the form

$$\mathbf{g}_{\mathbb{S}^n} = \frac{4}{\left(1 + |y|^2\right)^2} \mathbf{g}_{\mathbb{R}^n},$$

where $|y|^2 = \sum \left(y^i\right)^2$ and $\mathbf{g}_{\mathbb{R}^n} = \left(dy^1\right)^2 + \ldots + \left(dy^n\right)^2$ is the canonical Euclidean metric in $\mathbb{R}^n$.

(c) Prove that the Laplace operator $\Delta_{\mathbb{S}^2}$ on $\mathbb{S}^2$ has in the coordinates y^1, y^2 the form

$$\Delta_{\mathbb{S}^2} = \frac{\left(1 + |y|^2\right)^2}{4} \left(\frac{\partial^2}{\left(\partial y^1\right)^2} + \frac{\partial^2}{\left(\partial y^2\right)^2} \right).$$

3. Let M be a Riemannian manifold and μ be the Riemannian measure.

(a) Give the definition of the function spaces W^1, W_0^1, W_0^2 on M.

(b) Give the definition of the Dirichlet Laplace operator H on M as an operator in $L^2(M, \mu)$.

(c) Give a detailed proof of the fact that, for any $\alpha > 0$, the resolvent $R_\alpha := (H + \alpha \mathrm{id})^{-1}$ exists and is a bounded non-negative definite self-adjoint operator in $L^2(M, \mu)$. Show also that $\|R_\alpha\| \leq \alpha^{-1}$.

4. Let M be a Riemannian manifold and H be the Dirichlet Laplace operator in $L^2 = L^2(M, \mu)$, where μ is the Riemannian measure on M. Let $\Phi(\lambda)$ be a continuous real-valued function on $[0, +\infty)$ of subexponential growth; the latter means that, for any $\varepsilon > 0$,

$$\sup_{\lambda \in [0,\infty)} \left| \Phi(\lambda) e^{-\varepsilon \lambda} \right| < \infty. \tag{2}$$

(a) Prove that, for any $t > 0$, the operator

$$Q_t = \Phi(H) e^{-tH}$$

is a bounded self-adjoint operator in L^2. State clearly all the results used.

(b) A path $v(t) : (0, +\infty) \to L^2$ is said to satisfy the heat equation if, for any $t > 0$, $v(t) \in \operatorname{dom} H$, the Fréchet derivative $\frac{dv}{dt}$ exists, and

$$\frac{dv}{dt} = -Hv. \tag{3}$$

Prove that, for any $f \in L^2$, the path $v(t) = Q_t f$ satisfies the heat equation.

(c) Set $u(t) = \frac{dv}{dt}$ where $v(t)$ is as above. Prove that $u(t)$ also satisfies the heat equation.

5. Let M be a Riemannian manifold, H be the Dirichlet Laplace operator in $L^2 = L^2(M, \mu)$ (where μ is the Riemannian measure), and $P_t = e^{-tH}$ $(t \geq 0)$ be the heat semigroup.

(a) State without proof the main properties of the heat semigroup.

(b) Let ψ be a C^∞-function on $\mathbb{R}$ such that $\psi(0) = \psi'(0) = 0$ and $0 \leq \psi''(s) \leq 1$ for all s. Let f be an arbitrary function from L^2. Set $u_t = P_t f$ and prove that the following function

$$F(t) := \int_M \psi(u_t)\, d\mu \tag{4}$$

is continuous in $t \in [0, +\infty)$. State clearly any result used.

(c) Prove that the function $F(t)$ is differentiable for all $t > 0$ and that

$$F'(t) = \int_M \psi'(u_t) \frac{du_t}{dt} d\mu. \tag{5}$$

Hence, show that $F'(t) \leq 0$.

(d) Choosing a suitable function ψ in (4) and using the fact that the function $F(t)$ is decreasing, prove that $f \leq 1$ implies $u_t \leq 1$, for any $t > 0$.