UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M4P42/MSP12 ANALYSIS ON MANIFOLDS AND HEAT KERNELS

Date: Time:

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let M be an n–dimensional C^∞ manifold.

(i) State the definition of tangent bundle TM and give it the structure of a C^∞ vector bundle over M with fibre dimension n.

(ii) State the definition of Riemannian metric g on M.

(iii) Let M be equipped with a Riemannian metric g. Give coordinate independent definitions of the differential operators $\mathrm{grad}\colon C^\infty(M;\mathbb{R}) \to C^\infty(M;TM)$ and $\mathrm{div}\colon C^\infty(M,TM) \to C^\infty(M;\mathbb{R})$. Find their expressions in local coordinates $(x^1,\dots,x^n)$.

Hint: Utilise the fact that the Riemannian metric g induces a bundle isomorphism $TM \xrightarrow{\cong} T^*M$. Besides you can assume M be oriented if needed.

2. Let (M,g) be a C^∞ closed oriented Riemannian manifold.

(i) State a definition of the Sobolev spaces $H^s(M;\Lambda^pT^*M)$ for $s \geq 0$ in terms of the eigenvalues λ_j and eigenforms η_j for $j = 1,2,\dots$ of the Laplacian $-\Delta_p$ acting on differential p–forms.

(ii) Recall that the heat kernel $h_p(t,m,m')$ associated with the Laplacian $-\Delta_p$ acting on differential p–forms is defined by the property that $\omega = \omega(t,m)$ for $0 < t < \infty$, $m \in M$ given by

$$\omega(t,m) = \int_M h_p(t,m,m') \wedge *\omega_0(m'),$$

solves the heat equation

$$\frac{\partial\omega}{\partial t} - \Delta_p\omega = 0 \text{ in } (0,\infty)\times M$$

with initial condition $\omega(0,\cdot) = \omega_0 \in L^2(M;\Lambda^pT^*M)$. Here, $*\colon \Lambda^pT^*_mM \to \Lambda^{n-p}T^*_mM$ for $m \in M$ is the Hodge-$*$ operator; $n = \dim M$.

Provide an *informal* justification of the following formula for the heat kernel in terms of the eigenvalues λ_j and eigenforms η_j of $-\Delta_p$:

$$h_p(t,m,m') = \sum_{j=1}^\infty e^{-\lambda_j t}\eta_j(m)\otimes\eta_j(m').$$

Hint: 'Informal' means that questions about convergence of the infinite series (whatever the topology) need not be addressed.

3. Let M be a C^∞ closed manifold and let E, F be vector bundles over M.

(i) State the definition of differential operator $P \in \operatorname{Diff}^\mu(M; E, F)$ acting in sections of the vector bundles E, F.

(ii) State the definition of principal symbol $\sigma^\mu(P)$ of $P \in \operatorname{Diff}^\mu(M; E, F)$ and indicate to which space (of sections of a certain vector bundle) $\sigma^\mu(P)$ does belong.

(iii) Let $P \in \operatorname{Diff}^\mu(M; E, F)$ and

$$D(P) = \big\{u \in L^2(M; E);\ Pu \in L^2(M; F) \text{ in the weak sense}\big\}$$

be the maximal domain of P in $L^2(M; E)$. Verify that the operator $P\colon D(P) \subseteq L^2(M; E) \to L^2(M; F)$ is closed.

4. Let (M, g) be a C^∞ closed Riemannian manifold.

(i) State the definition of de Rham complex on M. What does it mean to say that the de Rham complex is a complex?

(ii) Let $\omega \in \Omega^* M = \bigoplus_{p=0}^n \Omega^p M$, where $n = \dim M$. Show that the following three conditions are equivalent:

(a) ω is a harmonic form, i.e., $\omega \in \ker \Delta$.

(b) $\omega \in \ker(d + \delta)$.

(c) ω is a closed form and $\delta\omega = 0$.

(iii) Show that the index of the de Rham–Hodge operator

$$d + \delta\colon \Omega^{\text{even}} M \to \Omega^{\text{odd}} M$$

equals the Euler characteristic $\chi(M) = \sum_{p=0}^n (-1)^p \dim H^p_{\text{dR}}(M)$. Recall that

$$\operatorname{ind}(d + \delta) = \dim \ker(d + \delta) - \dim \ker(d + \delta)^*,$$

while $\Omega^{\text{even}} M = \bigoplus_{p \text{ is even}} \Omega^p M$ and $\Omega^{\text{odd}} M = \bigoplus_{p \text{ is odd}} \Omega^p M$.

Hint: Make use of the Hodge theorem that relates $\dim H^p_{\text{dR}}(M)$ to the space of harmonic p–forms.

5. Let (M,g) be a C^∞ closed Riemannian manifold and let E, F be vector bundles over M; both equipped with bundle metrics. Let $P \in \mathrm{Diff}^1(M;E,F)$ be an elliptic differential operator.

(i) Show that the differential operator P^*P with domain $H^2(M;E)$ is selfadjoint in $L^2(M;E)$.

(ii) For $\lambda \in \mathbb{R}$, let $\mathcal{E}_\lambda = \{u \in C^\infty(M;E);\ P^*Pu = \lambda u\}$ and $\mathcal{F}_\lambda = \{u \in C^\infty(M;F);\ PP^*u = \lambda u\}$. Prove that:

(a) $\mathcal{E}_0 = \ker P$ and $\mathcal{F}_0 = \ker P^*$.

(b) For $\lambda \neq 0$, $P\colon \mathcal{E}_\lambda \to \mathcal{F}_\lambda$ is an isomorphism.

(iii) Provide short arguments for the following facts:

(a) Each space $\mathcal{E}_\lambda$, $\lambda \in \mathbb{R}$, is finite-dimensional.

(b) $\mathcal{E}_\lambda = \{0\}$ for all, but a discrete set of $\lambda \in \mathbb{R}$.