1. For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, define the mapping $m_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ by $m_\lambda(x) = \lambda x$.

(i) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Prove that $\widehat{\varphi \circ m_\lambda}(\xi) = \lambda^{-n} (\widehat{\varphi} \circ m_{\lambda^{-1}})(\xi)$ for all $\xi \in \mathbb{R}^n$.

(ii) Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Define distribution $u \circ m_\lambda$ by

$$(u \circ m_\lambda)(\varphi) = \lambda^{-n} u(\varphi \circ m_{\lambda^{-1}}),$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Prove that this definition is consistent with $\mathcal{S}(\mathbb{R}^n)$, i.e. show that if $u \in \mathcal{S}(\mathbb{R}^n)$, $(u \circ m_\lambda)(x) = u(\lambda x)$, and we identify u with its canonical distribution, then we have $(u \circ m_\lambda)(\varphi) = \lambda^{-n} u(\varphi \circ m_{\lambda^{-1}})$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(iii) Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Prove that $\widehat{u \circ m_\lambda} = \lambda^{-n} \widehat{u} \circ m_{\lambda^{-1}}$.

2. Let $f, g \in L^1(\mathbb{R}^n)$. Give the definition of the convolution $f * g$.

(i) Prove that $f * g \in L^1(\mathbb{R}^n)$. If we view $f * g$ as a tempered distribution, show that

$$(f * g)(\varphi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(z) g(y) \varphi(z + y) \, dy dz$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(ii) Let $f, g, h \in L^1(\mathbb{R}^n)$. Prove that $(f * g) * h \in L^1(\mathbb{R}^n)$ and that

$$((f * g) * h)(\varphi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(y) h(z) \varphi(x + y + z) \, dx dy dz$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(iii) Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Give the definition of the convolution $u * \varphi$. Prove that $u * \varphi \in C^\infty(\mathbb{R}^n)$.

3. Give the definition of the symbol classes S^m and $S^{-\infty}$. Give the definition of the pseudo-differential operator T_a with symbol $a \in S^m$.

Let $K(x, y)$ be the integral kernel of T_a, i.e. assume that $(T_a f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$.

(i) Prove that if $a \in S^{-\infty}$, then the kernel K of T_a satisfies

$$(1 + |x - y|)^N |\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_{\alpha\beta N}, \tag{1}$$

for some constants $C_{\alpha\beta N}$, and for all $x, y \in \mathbb{R}^n$, all multi-indices α, β, and all $N \geq 0$.

(ii) Conversely, assume that the kernel K of a pseudo-differential operator T_a satisfies estimate (1) for all $x, y \in \mathbb{R}^n$, all multi-indices α, β, and all $N \geq 0$. Prove that then $a \in S^{-\infty}$.

4. Let $T_a \in \Psi^m$ be a pseudo-differential operator of order m.

(i) Give the definition of the L^2-adjoint T_a^* of $T_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Consequently, for $u \in \mathcal{S}'(\mathbb{R}^n)$, give the definition of $T_a u$ using the L^2-adjoint T_a^*.

(ii) Check that the definition of $T_a u$ in (i) is consistent with the standard definition if $u \in \mathcal{S}(\mathbb{R}^n)$, i.e. show that $\int_{\mathbb{R}^n} T_a u(x)\varphi(x)dx = \int_{\mathbb{R}^n} u(x)\overline{T_a^*\overline{\varphi}(x)}dx$ for all $u, \varphi \in \mathcal{S}(\mathbb{R}^n)$.

(iii) For $u \in \mathcal{S}'(\mathbb{R}^n)$, define *the transpose* T_a^t of T_a by the formula

$$T_a^t u := \overline{T_a^* \overline{u}}.$$

Here $\overline{u} \in \mathcal{S}'(\mathbb{R}^n)$ is the complex conjugate of u, defined by $\overline{u}(\varphi) = \overline{u(\overline{\varphi})}$, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $\overline{u(\overline{\varphi})}$ is the complex conjugate of $u(\overline{\varphi}) \in \mathbb{C}$.
Prove that

$$(T_a^*\overline{u})(\psi) = \overline{u(\overline{T_a \overline{\psi}})},$$

for all $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then also prove that

$$(T_a^t u)(\varphi) = u(T_a \varphi)$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(iv) Prove that T_a^t defined in (iii) is a pseudo-differential operator, with symbol having an asymptotic expansion

$$\text{symbol of } T_a^t \sim \sum_\alpha \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \left[a(x, -\xi)\right].$$

[Here you may use the theorem on compound symbols without proof]

5. For a function $\psi \in C_0^\infty(\mathbb{R}^n)$, give the definition of its support $\operatorname{supp}\psi$.

(i) For $\psi \in C_0^\infty(\mathbb{R}^n)$, let us define operator $M : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by setting $(Mf)(x) = \psi(x)f(x)$, for all $x \in \mathbb{R}^n$. Regarding M as a pseudo-differential operator, calculate its order and symbol.

(ii) Let $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\varphi \bigcap \operatorname{supp}\psi = \emptyset$. Let T_a be a differential operator of order m (i.e. assume that its symbol $a = a(x, \xi)$ is a polynomial in ξ or order m). Prove that $(\varphi T_a(\psi f))(x) = 0$ for all $f \in C^\infty(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$.

(iii) Let $a \in S^m$ be now a general symbol of order m. Let φ, ψ be as in (ii). Prove that operator R defined by $(Rf)(x) = (\varphi T_a(\psi f))(x)$ is a pseudo-differential operator of order $-\infty$.
[Here you may use the composition formula for pseudo-differential operators without proof]