Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MSc EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M4P41/MSP11

Analytic methods in partial differential equations

Date: Tuesday, 1 June 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. State the Riemann-Lebesgue lemma.

(i) Suppose that $u, f \in L^1(\mathbb{R}^n)$ satisfy $(1-\Delta)u = f$. Prove that $\widehat{u}(0) = \widehat{f}(0)$ and hence $\int_{\mathbb{R}^n} u(x)dx = \int_{\mathbb{R}^n} f(x)dx$.

(ii) Let $u, f \in L^1(\mathbb{R}^n)$ satisfy $(1-\Delta)u = f$. Suppose that f satisfies

$$|\widehat{f}(\xi)| \leq \frac{C}{(1+|\xi|)^{n-1}}, \quad \text{for all } \xi \in \mathbb{R}^n.$$

Prove that u is a bounded continuous function on $\mathbb{R}^n$.

(iii) Suppose that $u \in L^1(\mathbb{R}^n)$ satisfies $(1-\Delta)u = \delta$, where δ is the standard δ-distribution defined by $\delta(\phi) = \phi(0)$, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. Prove that $\int_{\mathbb{R}^n} u(x)dx = 1$.

2. Give definitions of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and of the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions.

(i) Show that $L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$. Show that if $f_k \to f$ in $L^p(\mathbb{R}^n)$, then $f_k \to f$ in $\mathcal{S}'(\mathbb{R}^n)$.

(ii) Let $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Prove that

$$\int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}dx = \int_{\mathbb{R}^n} \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi.$$

(In this part you may use the Fourier inversion formula without justification, if necessary.)

(iii) Let $u \in L^2(\mathbb{R}^n)$. Prove that $\widehat{u} \in L^2(\mathbb{R}^n)$ and that $||u||_{L^2(\mathbb{R}^n)} = ||\widehat{u}||_{L^2(\mathbb{R}^n)}$.
(In this part you may assume properties of $L^2(\mathbb{R}^n)$ without proof.)

3. For $x \in \mathbb{R}$, let us define $\chi(x) = 0$ for $x \leq -1$, $\chi(x) = -1$ for $-1 < x \leq 0$, $\chi(x) = 1$ for $0 < x \leq 1$, and $\chi(x) = 0$ for $1 < x$.

(i) Prove that $\widehat{\chi}(\xi) = \frac{1}{\pi i \xi}(1 - \cos(2\pi\xi))$.

(ii) For $a \in \mathbb{R}$, define $\delta_a(\phi) = \phi(a)$, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. Prove that $\widehat{\delta_a}(x) = e^{-2\pi i a x}$.

(iii) Prove that $\chi' = -\delta_1 + 2\delta_0 - \delta_{-1}$.

(iv) Prove that $\widehat{\chi'}(\xi) = 2 - 2\cos(2\pi\xi)$.

4. Give the definition of the symbol class S^m and of a pseudo-differential operator with symbol $a \in S^m$.

   Let $T_a \in \Psi^m$.

   (i) Prove that T_a maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. In other words, show that if $f \in \mathcal{S}(\mathbb{R}^n)$, then $T_a f \in \mathcal{S}(\mathbb{R}^n)$.

   (ii) Derive the formula for the adjoint operator T_a^*.

   (iii) Assume that $a = a(x, \xi) \in S^0$ is compactly supported with respect to x. Prove that $T_a : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous.

5. For $u \in C(\mathbb{R}^n)$ give the definition of the support of u.

   Let A be a linear differential operator

$$Af(x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha f(x)$$

   with coefficients $a_\alpha \in C^\infty(\mathbb{R}^n)$, $|\alpha| \leq m$.

   (i) Prove that $\operatorname{supp} Af \subset \operatorname{supp} f$, for all $f \in C^\infty(\mathbb{R}^n)$.

   (ii) Let T be an operator defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

   with $K \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.
   Prove that T defines a (sequentially) continuous operator from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ and from $S'(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$.

   (iii) For operators T from (ii) with $K \not\equiv 0$, show that we can never have the property $\operatorname{supp} Tf \subset \operatorname{supp} f$ for all $f \in C^\infty(\mathbb{R}^n)$.