Imperial College
London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

and MSc EXAMINATIONS

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M4P41/MSP11 Analytic Methods in Partial Differential Equations

Date: Tuesday 24th May 2005 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Prove that the Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ continuously.

Let $u \in C(\mathbb{R}^n)$ satisfy $|u(x)| \leq C\langle x\rangle^N$ for some constants C, N, where (as usual) $\langle x\rangle = (1+|x|^2)^{1/2}$. Let $k > N+n$. Let us define

$$v_k(\phi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} u(x)\langle x\rangle^{-k}\langle D_\xi\rangle^k \phi(\xi)\,dx\,d\xi,$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$.

(ii) Prove that $v_k \in \mathcal{S}'(\mathbb{R}^n)$.

(iii) Prove that there is $v \in \mathcal{S}'(\mathbb{R}^n)$ such that $v = v_k$ for all $k > N+n$. Show that $v = \hat{u}$.

2. (i) State Hölder's inequality and prove it. You may use Young's inequality without proof here.

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 0$ for $|x| \geq 1$ and $\int_{\mathbb{R}^n}\eta(x)dx = 1$. Let $\epsilon > 0$ and define $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$. Let $1 \leq p \leq \infty$ and $f \in L^1_{loc}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Define $f_\epsilon = f * \eta_\epsilon$.

(ii) Prove that $f_\epsilon \in C^\infty(\mathbb{R}^n)$.

(iii) Prove that $||f_\epsilon||_{L^p} \leq ||f||_{L^p}$.

(iv) Suppose further that $f \in C(\mathbb{R}^n)$ and that f is compactly supported. Prove that $f_\epsilon \to f$ as $\epsilon \to 0$, uniformly on compact sets.

3. (i) Give the definition of the Sobolev space $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$.

(ii) Let $u \in H^s(\mathbb{R}^n)$ with some $s > n/2$. Prove that $\hat{u} \in L^1(\mathbb{R}^n)$.

(iii) Let $s > n/2$. Using (ii), prove that there is a constant $C > 0$ such that we have

$$\sup_{x\in\mathbb{R}^n} |u(x)| \leq C||u||_{H^s(\mathbb{R}^n)}$$

for all $u \in C_0^\infty(K)$ for all compact sets $K \subset \mathbb{R}^n$.

(iv) Prove that $\bigcap_s H^s_{loc}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

4. Let

$$Au(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} a(x,y,\xi)u(y)dyd\xi \tag{1}$$

be an operator with a compound symbol $a \in \widetilde{S^m}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$.

(i) Viewing A as a pseudo-differential operator, write down the asymptotic expansion for its symbol $\sigma_A(x,\xi)$ in terms of the compound symbol a. Define what this asymptotic expansion means.

(ii) Prove that if u is smooth in some neighborhood of a point $x \in \mathbb{R}^n$, then Au is also smooth in some neighborhood of the same point x.

(iii) Let $Bu(x) = \int_{\mathbb{R}^n} K(x,y)u(y)dy$ be an operator from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$ with kernel $K(x,y) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Show that B is an operator with compound symbol and it can be written in the form (1) with some $a \in \widetilde{S^{-\infty}}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$.

5. Let $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, $a_\alpha \in C^\infty(\mathbb{R}^n)$, and let $p(x,D)$ be the differential operator with symbol $p(x,\xi)$.

(i) Prove that

$$p(x,D)(f(x)g(x)) = \sum_\alpha \frac{1}{\alpha!}[p^{(\alpha)}(x,D)f(x)][D^\alpha g(x)]$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, where $p^{(\alpha)}(x,D)$ is a differential operator with symbol $p^{(\alpha)}(x,\xi) = \partial_\xi^\alpha p(x,\xi)$.

(ii) Explain how to deduce the composition formula for differential operators from (i).

(iii) Let $p(x,D) \in \Psi^m$ be now a general pseudo-differential operator of order m. Let $f \in L^2(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Prove that

$$p(x,D)(f(x)g(x)) - \sum_{|\alpha|\le m} \frac{1}{\alpha!}[p^{(\alpha)}(x,D)f(x)][D^\alpha g(x)] \in H^1(\mathbb{R}^n).$$