Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MSc EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M4P38/MSP8

Geometric and Combinatorial Group Theory

Date: Monday, 22nd May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Define what it means to say that "$F = F(S)$ is a free group on the set S".

   (b) Define the *rank* of a finitely generated free group, and prove that it is well-defined.

   (c) Describe three subgroups of index 3 in $F_2 = F(\{a, b\})$, and give a basis for each.

   (There are more than three such subgroups, but you do not need to find them all. However, you must show that no pair of your examples are the same by giving elements that are in one subgroup but not in the other.)

   [Hint: Consider covering-spaces of the graph with one vertex and two edges.]

2. (a) Define what it means to say that $\langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$ is an abstract presentation of the group Γ.

   (b) Prove that the group $\langle x, y \mid x^{-1}yxy^{-2},\, y^{-1}xyx^{-2} \rangle$ is trivial.

   (c) Make the group $D := \langle a, b \mid a^2,\, b^2 \rangle$ act by isometries on $\mathbb{R}$, with a and b acting as reflections in distinct points.

   [You should give a formula for the action and prove that it is well-defined.]

   (d) Deduce that ab generates an infinite subgroup of D.

   (e) Prove that $ab = (ba)^{-1}$ in D.

   (f) Prove that every element in D is equal to a *unique* word of the form $(ab)^n a^i$ with $n \in \mathbb{Z}$ and $i \in \{0, 1\}$.

3. (a) Define the free product $A * B$ of groups A and B.

(b) Define the Cayley graph $\mathcal{C}_S(G)$ of a group G with finite generating set S.

(c) Sketch the Cayley graph of $G = \langle a, b \mid a^2, b^3 \rangle$ with $S = \{a, b\}$.

There is a normal form theorem (often called Britton's Lemma) that says certain words in the generators $A \cup \{t\}$ are not equal to 1 in $G*_\theta$. In what follows you may use (without proof) the following consequence of Britton's Lemma:
The subgroup of

$$B = \langle a, t \mid ta^3t^{-1}a^{-4} \rangle$$

generated by $c := [tat^{-1}, a]$ and t is a free subgroup of rank 2.

(d) Prove that the homomorphism $\phi : B \to B$ defined by $\phi(a) = a^3$, $\phi(t) = t$ is surjective but has c in its kernel.

(e) Prove that if Q is finite then for any homomorphism $\pi : B \to Q$ one has $\pi(c) = 1$.
[Hint: Choose $c_n \in B$ with $\phi^n(c_n) = c$. Let $\psi_n := \pi \circ \phi^n$. By considering $\psi_m(c_n)$, argue that if $\pi(c) \neq 1$ then all of the maps ψ_n would be distinct.]

(f) Deduce that the amalgamated free product

$$B *_{F_2} \widehat{B} = \langle a, t, \widehat{a}, \widehat{t} \mid ta^3t^{-1}a^{-4},\ \widehat{t}\widehat{a}^3\widehat{t}^{-1}\widehat{a}^{-4},\ c^{-1}\widehat{t},\ t^{-1}\widehat{c} \rangle$$

has no non-trivial finite quotients.
(Here, the F_2 in the amalgamation is generated by c, t on the side B, and these generators are identified with $\widehat{t}, \widehat{c}$ (order reversed) on the $\widehat{B}$ side.)
[Hint: Note that $\langle a, t \mid ta^3t^{-1}a^{-4},\ t \rangle = \{1\}$, so if t lies in the kernel of any homomorphism from B, then a does too.]

4. (a) Let N and Q be groups and let $\Phi : Q \to \mathrm{Aut}(N)$ be a homomorphism (i.e., an action of Q on N). Describe the associated semi-direct product $N \rtimes_\Phi Q$.

   (b) Let $H = \langle x,\, y,\, z \mid xyx^{-1}y^{-1} = z,\ [x, z] = [y, z] = 1\rangle$.
   Explain why every element of H can be written in the form $z^n y^m x^p$ with $n, m, p \in \mathbb{Z}$.

   (c) Let G be the group of 3×3 integer matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

   You may assume that G is generated by elementary matrices. (Recall that an elementary matrix E_{ij} has a 1 in each diagonal place and zeros everywhere else except for a 1 in the ij-place.)
   Prove that H is isomorphic to G.
   [Hint: Send x, y, z to E_{12}, E_{23}, E_{33} respectively; check that this is a homomorphism; use part (b) to see that it is injective.]

   (d) Let ϕ be the automorphism of $\mathbb{Z}^2 = \langle Y, Z \mid [Y, Z]\rangle$ defined by $\phi(Y) = YZ$, $\phi(Z) = Z$.
   Let $P = \mathbb{Z}^2 \rtimes_\Phi \mathbb{Z}$, where Φ sends the generator of $\mathbb{Z}$ to ϕ.
   Prove that $H \cong P$.

5. (a) Define the terms "simply connected" and "contractible".

   (b) State the Seifert-van Kampen Theorem. (You do not need to define pushouts.)

   (c) Prove that if $n \geq 2$ then $\pi_1 \mathbb{S}^n$ is trivial.

   (d) Define the connected sum of two connected, n-dimensional topological manifolds, and describe (without proof) its fundamental group in the case $n \geq 3$.

   (e) Sketch a compact, 2-dimensional complex K with fundamental group

$$\langle a,\, b,\, c \mid aba^{-1}b^{-1}\rangle.$$

   Also sketch the covering space corresponding to the index-2 subgroup

$$\langle a, b, c^2, cac^{-1}, cbc^{-1}\rangle.$$