Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

and MSc EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M4P38/MSP8 Geometric and Combinatorial Group Theory

Date: 25 May 2005 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Define what it means to say that "$F = F(S)$ is a free group, freely generated by the set S".

   (b) Let S be a set. Assuming that there exists a free group freely generated by S, prove that it is *unique* up to canonical isomorphism.

   (c) Define the *rank* of a finitely generated free group, and prove that it is well-defined.

   (d) Prove that $H = \langle a, b, c \mid [a, b] = c,\ [a, c] = [b, c] = 1 \rangle$ does not contain a non-abelian free group. [Hint: growth]

2. (a) Define what it means to say that $\langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$ is an abstract presentation of the group Γ.

   (b) State (but do not prove) Tietze's Theorem.

   (c) Sketch the proof that every finitely presented group is the fundamental group of a compact 2-dimensional complex.

   (d) Describe and sketch a 2-dimensional space whose fundamental group is $\langle a, b, c, d \mid aba^{-1}b^{-1},\ c \rangle$.

3. (a) Define what it means to say that a map between topological spaces $p : \widehat{X} \to X$ is a *covering map*.

   (b) Let X be a connected, locally-connected topological space with basepoint x_0 and let $H \subset \pi_1(X, x_0)$ be a subgroup.
   Explain briefly how to obtain a covering space $\widehat{X} \to X$ with $\pi_1(\widehat{X}) \cong H$.

   (c) Describe (without proof) the fundamental group of a compact graph.

   (d) Define the Euler characteristic $\chi(Y)$ of a compact graph Y and explain why if $p : \widehat{Y} \to Y$ is a d-sheeted covering, $\chi(\widehat{Y}) = d \cdot \chi(Y)$.

   (e) Noting that graphs of the same Euler characteristic have isomorphic fundamental groups, prove that if F is a free group of rank r and $H \subset F$ is a subgroup of index k, then H is free of rank $k(r-1)+1$.

   (f) Draw a 2-sheeted covering space of the graph with 1 vertex and 3 edges.

4. (a) Define HNN extension and amalgamated free products.

   (b) Prove that the rationals $\mathbb{Q}$, with the operation of addition, is *not* a finitely generated group but is a subgroup of a finitely generated group.

   (c) Let $B = \langle a, t \mid ta^2t^{-1}a^{-3} \rangle$. Prove that the homomorphism $\phi : B \to B$ defined by $\phi(a) = a^2$, $\phi(t) = t$ is surjective but is not an isomorphism.

   (d) A group G is said to be *residually finite* if for every $g \in G$ with $g \neq 1$, there exists a homomorphism $\pi : G \to Q$, so that Q is a finite group and $\pi(g) \neq 1$.

   Prove that B is not isomorphic to a subgroup of a residually finite group.

5. (a) Define the fundamental group $\pi_1(X, p)$ of a connected, locally simply connected space in terms of loops in X.

   (b) Define the terms "simply connected" and "contractible".

   (c) Let A be a group with presentation $\langle S_A \mid R_A \rangle$ and let B be a group with presentation $\langle S_B \mid R_B \rangle$. Let C be a group with generating set T. Define the push-out of the diagram of groups

$$A \overset{f_A}{\leftarrow} C \overset{f_B}{\rightarrow} B$$

   and give (without proof) a presentation of it.

   (d) State the Seifert-van Kampen Theorem.

   (e) Prove that if $n \geq 2$ then $\pi_1(\mathbb{S}^n, x)$ is trivial.

   (f) What is the fundamental group of $(\mathbb{S}^1 \times \mathbb{S}^3)\#(\mathbb{S}^1 \times \mathbb{S}^3)$? [no proof required]