Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MSc EXAMINATIONS (MATHEMATICS) May-June 2006

This paper is also taken for the relevant examination for the Associateship.

# M4P36/MSP6

## Representation Theory of Symmetric Groups

Date: Wednesday, 24th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

 

*Throughout this paper, $n$ is a non-negative integer, $\mathfrak{S}_n$ is the symmetric group of degree $n$, and $F$ is a field.*

1. Let $\lambda$ be a partition of $n$. Explaining all the terms you use, define the permutation module $M^\lambda$ for $F\mathfrak{S}_n$, and define a symmetric, $\mathfrak{S}_n$-invariant, non-singular bilinear form on $M^\lambda$. Define, also, the Specht module $S^\lambda$.

   Now let $p$ be a prime number and assume that $F$ has characteristic $p$. State what it means to say that $\lambda$ is $p$-singular. Prove that if $\lambda$ is $p$-singular, then $S^\lambda \subseteq S^{\lambda\perp}$. What can be said about $S^\lambda/(S^\lambda \cap S^{\lambda\perp})$ when $\lambda$ is $p$-regular?

   Prove that the number of $p$-regular partitions of $n$ equals the number of $p$-regular conjugacy classes of $\mathfrak{S}_n$.

2. For a given $\lambda$-tableau $t$, the Garnir relations state that certain elements of the group algebra $F\mathfrak{S}_n$ annihilate the generator $e_t$ of the Specht module $S^\lambda$. State and prove the Garnir relations.

   What other elements of the group algebra annihilate $e_t$?

   Explain what is meant by the standard basis of $S^\lambda$.

   Now let $\lambda = (2, 2, 1)$ and

$$t = \begin{matrix} 2 & 1 \\ 3 & 5 \\ 4 & \end{matrix} \quad .$$

   Express $e_t$ as a linear combination of the elements of the standard basis of $S^\lambda$.

3. Define what is meant by saying that $\lambda$ dominates $\mu$ for partitions $\lambda$ and $\mu$ of $n$. State and prove the Basic Combinatorial Lemma which relates the dominance order to a property of tableaux.

   Suppose that $S^\lambda$ is a composition factor of the permutation module $M^\mu$, defined over a field of characteristic zero. What relationship can you deduce between $\lambda$ and $\mu$? Justify your answer. State Young's Rule which determines the composition factors of $M^\mu$ precisely.

4. State the Murnaghan-Nakayama Rule for determining the values of the irreducible characters $\chi^\lambda$ of $\mathfrak{S}_n$.

Let $\phi$ and $\psi$ be the following characters of $\mathfrak{S}_8$ :

$$\phi = \chi^{(8)} + \chi^{(6,2)} + \chi^{(4,4)}, \qquad \psi = \chi^{(7,1)} + \chi^{(5,3)}.$$

(a) Prove that $\phi$ and $\psi$ are equal on all permutations whose cycle decomposition contains a cycle of odd length.

(b) Find the values of $\phi$ and $\psi$ on the identity permutation.

(c) Find the values of $\phi$ and $\psi$ on an $8$-cycle.

5. Let $\lambda$ be a partition of $n$. State the Branching Theorem which determines the restriction $\chi^\lambda \downarrow \mathfrak{S}_{n-1}$ of the irreducible character $\chi^\lambda$ of $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$.

For which partitions $\lambda$ is $\chi^\lambda \downarrow \mathfrak{S}_{n-1}$ an irreducible character of $\mathfrak{S}_{n-1}$? Using this information, or otherwise, prove that for $n \geq 5$, every irreducible character of $\mathfrak{S}_n$ (except $\chi^{(n)}$ and $\chi^{(1^n)}$) has degree at least $n-1$.

(*The degree of a character is its value on the identity element. You may assume that the result holds for* $n=5$ *and for* $n=6$.)