1. Let R be a ring and let M be a left R-module. Define the terms

(a) M is *irreducible*;

(b) a *composition series* for M.

Show that the following statements are equivalent,

(i) M is both Artinian and Noetherian.

(ii) Any proper chain of submodules of M can be refined to a composition series.

(iii) M has a composition series.

(You may use without proof standard facts about Noetherian modules and Artinian modules.)

Write down a composition series for the matrix ring $M_r(D)$ where D is a division ring. (A detailed proof is not required.)

2. Let R be a ring and let M be a left R-module. Define the terms

(a) M is *semisimple*;

(b) M is *completely reducible*.

State the Complementation Lemma, and use it to deduce that the following statements are equivalent.

(i) M is Artinian semisimple.

(ii) $M = L_1 \oplus \cdots \oplus L_k$, where L_i is irreducible, $i = 1, \ldots, k$, and k is an integer.

Let M be a left module over a division ring D. Show that M has a basis.

Give, with brief explanations, examples of

(1) a module that is semisimple but not Artinian;

(2) a module that is Artinian but not semisimple.

3. Let $R = R_1 \times \cdots \times R_k$ be a direct product of a finite set of rings $R_1, \ldots, R_k$. Write down the corresponding orthogonal central idempotents of R, and explain why they have the required properties. (You are *not* required to show that R is a ring.)

Show that if I is an irreducible R_i-module for some i, then I is also an irreducible R-module. Prove also the converse.

Suppose that I is an irreducible R_i-module and that J is an irreducible R_j-module for some $j \neq i$. Show that $I \ncong J$ as an R-module.

Suppose a ring R is left Artinian and left semisimple. State the Wedderburn-Artin Theorem, and use it to give a list of irreducible left R-modules such that no two members of the list are isomorphic, but any irreducible left R-module is isomorphic to a member of the list.

4. Let R be a ring, M a left R-module. Define the *radical* $\mathrm{rad}(M)$ of M.

Let $\alpha : M \to N$ be a homomorphism of left R-modules. Show that $(\mathrm{rad}(M))\alpha \subseteq \mathrm{rad}(N)$. Deduce that $\mathrm{rad}(R)$ is a twosided ideal of R.

Suppose that R is left Artinian. Show that $\mathrm{rad}(R)$ is the maximal twosided nilpotent ideal of R. (You may quote results from Nakayama's Lemma as required.)

Find the radicals of the following rings.

(1) $\mathbb{Z}/\mathbb{Z}a,\ a > 1$;

(ii) $T = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ where D is a division ring.

5. Let R be an integral domain. State the right Ore condition for $a, b \in R$, $b \neq 0$.

Define a relation $\backsim$ on $R \times R^*$ by $(a, b) \backsim (c, d)$ if there exist nonzero $u, v \in R$ with $au = cv$, $bu = dv$. Given that $\backsim$ is an equivalence relation, define the "right fraction" ab^{-1}, and show how the Ore condition is used to define addition and multiplication on the set Q of all such fractions. (You are *not* required to verify that this addition and multiplication are well-defined or that Q is a ring.)

Suppose that an integral domain R contains two nonzero elements a, b with $aR \cap bR = 0$. Show that R contains a direct sum $aR \oplus baR \oplus \cdots \oplus b^i aR$ for any $i > 1$.

Deduce that a right Noetherian integral domain satisfies the right Ore condition.

Is the converse true?