1. Let Γ_1 be a graph whose vertex-set is $\{a_i, b_j \mid 0 \leq i, j \leq 4\}$, where a_i and a_k are adjacent iff $i - k \equiv \pm 1 \bmod 5$; b_j and b_k are adjacent iff $j - k \equiv \pm 2 \bmod 5$; a_i and b_j are adjacent iff $i = j$. Let Γ_2 be a graph whose vertex-set is $\{c_i, d_j, e \mid 0 \leq i \leq 5, 0 \leq j \leq 2\}$, where c_i and c_k are adjacent iff $i - k \equiv \pm 1 \bmod 6$; c_i and d_j are adjacent iff $i \equiv j \bmod 3$; e is adjacent to d_0, d_1, d_2 and there are no further adjacencies.

Prove that Γ_1 and Γ_2 are isomorphic graphs.

Let Γ be a graph isomorphic to Γ_1 and Γ_2 and let A be the automorphism group of Γ.

Prove the following assertions:

(i) A contains the Frobenius group F of order 20, where

$$F = \langle \rho, \sigma \mid \rho^5 = \sigma^4 = 1, \sigma^{-1}\rho\sigma = \rho^2 \rangle;$$

(ii) for any two vertices v, u of Γ there is an element in F (and hence also in A) which sends v to u;

(iii) for a vertex v of Γ the subgroup of A formed by the automorphisms which send v to v is isomorphic to the dihedral group D_{12} of order 12;

(iv) $|A| = 120$.

Let $GL_n(\mathbb{C})$ be the group of invertible $n \times n$ matrices with complex entries with respect to the matrix multiplication. With A as above let $\varphi : A \to GL_n(\mathbb{C})$ be a faithful representation (a group homomorphism with trivial kernel).

Prove that $n \geq 4$.

Turn over...

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2. Let Ω be a set of n elements, where $n \geq 5$, let $S_n = \mathrm{Sym}(\Omega)$ be the symmetric group on Ω, let t be an element of S_n acting on Ω as a transposition (so that $t = (a, b)$ for some $a, b \in \Omega$), and let α be an automorphism of S_n.

Making use of the following Coxeter presentation of S_n (which you don't have to prove):

$$S_n = \langle t_1, ..., t_{n-1} \mid t_i^2 = 1, (t_i t_j)^{m_{ij}} = 1 \text{ for } 1 \leq i < j \leq n-1 \rangle$$
$$\text{where } m_{i\,i+1} = 3 \text{ and } m_{ij} = 2 \text{ if } j - i \geq 2,$$

prove the following assertions:

(i) if $\alpha(t)$ acts on Ω as a transposition then α is an inner automorphism of S_n (so that there is $g \in S_n$ such that $\alpha(h) = g^{-1}hg$ for every $h \in S_n$);

(ii) there exists an automorphism β of S_6 which is not an inner automorphism;

(iii) the automorphism β in (ii) restricts to an automorphism on the alternating group A_6 which is not an inner automorphism of A_6.

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3. Let n be a positive integer, and let $\mathbb{R}^n = \{x = (x_1, ..., x_n) \mid x_i \in \mathbb{R}\}$ be an n-dimensional real vector space with the *standard* inner product $(x, y) = \sum_{i=1}^n x_i y_i$. With a non-zero vector $r \in \mathbb{R}^n$ associate the *reflection* ρ_r which is a linear transformation of $\mathbb{R}^n$ which sends r to its negative while fixing every vector in the hyperplane $H_r = \{x \mid x \in \mathbb{R}^n, (r, x) = 0\}$. Let $r, s \in \mathbb{R}^n$, let $(r, r) = (s, s) = 1$, and let $\tau = \rho_s \rho_r$ be the product of the reflections associated with r and s.

Describe in terms of the inner product (r, s) the necessary and sufficient condition for the order of τ to be finite.

Let S_n be the symmetric group of $\{1, ..., n\}$. Define the action of S_n on $\mathbb{R}^n$ by the following rule: $g(x) = (x_{g^{-1}(1)}, ..., x_{g^{-1}(n)})$ where $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $g \in S_n$ and $g^{-1}(i)$ is the image of i under the inverse of g, where $1 \le i \le n$.

Prove the following assertions:

(i) the above defined action is a faithful representation $\varphi : S_n \to GL_n(\mathbb{R})$ and $\varphi(S_n)$ preserves the standard inner product in the sense that $(x, y) = (g(x), g(y))$ for all $x, y \in \mathbb{R}^n$ and all $g \in S_n$;

(ii) if t is a transposition of S_n then there exists $r(t) \in \mathbb{R}^n$ such that $\varphi(t) = \rho_{r(t)}$;

(iii) the hyperplanes $H_{r(t)}$ taken for all the transpositions t in S_n share a 1-dimensional subspace.

Calculate the order of the automorphism group of the dihedral group D_8 of order 8.

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4. For $\mathbb{R}^8$ being an 8-dimensional real vector space with the standard inner product, let Λ be the set of vectors $\lambda = (\lambda_1, ..., \lambda_8) \in \mathbb{R}^8$ satisfying the following:

(a) $2\lambda_i \in \mathbb{Z}$ and $2\lambda_i \equiv 2\lambda_j \bmod 2$ for all $1 \leq i \leq j \leq 8$;

(b) $\sum_{i=1}^{8} \lambda_i \in 2\,\mathbb{Z}$.

Prove the following assertions:

(i) $m\lambda + n\mu \in \Lambda$ for all $\lambda, \mu \in \Lambda$ and all $m, n \in \mathbb{Z}$;

(ii) $(\lambda, \lambda) \in 2\,\mathbb{Z}$ for all $\lambda \in \Lambda$;

(iii) $(\lambda, \mu) \in \mathbb{Z}$ for all $\lambda, \mu \in \Lambda$;

(iv) if $x \in \mathbb{R}^8$ and $(x, \lambda) \in \mathbb{Z}$ for all $\lambda \in \mathbb{Z}$ then $x \in \Lambda$.

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5. Follow the notation introduced in the first paragraph of Question 4.

Calculate the size of $\Lambda_2 = \{\lambda \in \Lambda \mid (\lambda, \lambda) = 2\}$.

Prove the following assertions:

(i) Λ_2 contains a basis of $\mathbb{R}^8$;

(ii) if $\lambda, \mu \in \Lambda_2$ and $(\lambda + \mu)/2 \in \Lambda$ then $\lambda = \pm\mu$;

An automorphism g of Λ is a linear transformation of $\mathbb{R}^8$ which preserves Λ as a set and also preserves the standard inner product (in the obvious sense that $(x, y) = (g(x), g(y))$ for all $x, y \in \mathbb{R}^8$). For $\varepsilon = (\varepsilon_1, ..., \varepsilon_8)$ with $\varepsilon_i \in \{\pm 1\}$, let g_ε denote the transformation of $\mathbb{R}^8$ such that $g_\varepsilon(x) = (\varepsilon_1 x_1, ..., \varepsilon_8 x_8)$ for $x = (x_1, ..., x_8)$.

Prove the following assertions:

(iii) g_ε is an automorphism of Λ if and only if $\Pi_{i=1}^8 \varepsilon_i = 1$;

(iv) if g is an automorphism of Λ such that $g(\lambda) = \lambda$ for every $\lambda \in \Lambda_2$ then g is the identity automorphism;

(v) if g is an automorphism of Λ such that $(g(\lambda) + \lambda)/2 \in \Lambda$ for every $\lambda \in \Lambda$ then $g = \pm I$, where I is the identity operator.