1. (i) Define a projective curve in $\mathbb{CP}^2$ and explain why the definition gives a well-defined subset of $\mathbb{CP}^2$. What condition should the defining polynomial of a projective curve satisfy so that the polynomial is uniquely determined by the subset of $\mathbb{CP}^2$ (up to scalar multiplication)?

(ii) Define irreducible and reducible projective curves.

(iii) Define a singular point of a projective curve C. Prove that any reducible projective curve C in $\mathbb{CP}^2$ has at least one singular point.

(iv) Prove that any irreducible cubic projective curve C in $\mathbb{CP}^2$ has at most 1 singular point. (You may use any standard facts about the relation between intersection multiplicities and the multiplicities of singular points provided you state them clearly).

2. Consider the following family of complex cubics defined on $\mathbb{C}^3$

$$P_{\mu,\lambda} = \mu(x^3 + y^3 + z^3) + 3\lambda xyz,$$

where μ and λ are complex numbers, not both zero.

(i) Prove that if the projective curve $C_{\mu,\lambda}$ defined by $P_{\mu,\lambda}$ is singular then $\mu = 0$ or $\left(\frac{\lambda}{\mu}\right)^3 = -1$.

(ii) If $\mu = 0$ find the singular points of $C_{0,\lambda}$ and show that $C_{0,\lambda}$ is a union of three distinct lines (which you should explicitly identify). What is the relationship between the geometry of these 3 lines and the singular points of $C_{0,\lambda}$?

(iii) If $C_{\mu,\lambda}$ is singular and $\mu \neq 0$, prove that at any singular point $p = [x, y, z]$, the three ratios $\frac{x}{y}, \frac{x}{z}, \frac{y}{z}$ are all cube roots of 1. Hence or otherwise, find the three singular points of $C_{\mu,\lambda}$ in the case where $\left(\frac{\lambda}{\mu}\right) = -\exp(\frac{2\pi i}{3})$.

(iv) By considering lines passing through these singular points show that when $\left(\frac{\lambda}{\mu}\right) = -\exp(\frac{2\pi i}{3})$, $C_{\mu,\lambda}$ is a union of three distinct lines, which you should find.

3. Let C be a projective curve in $\mathbb{CP}^2$ defined by the homogeneous polynomial P of degree d.

(i) Define an inflection point of C.

(ii) Prove that if $d \geq 3$, and C is nonsingular then it has at least one and at most $3d(d-2)$ points of inflection. (You may assume that every point of an irreducible projective curve of degree d is an inflection point if and only if $d = 1$.)

(iii) Suppose that C is nonsingular and does not contain the point $[0, 1, 0]$. Define the map $\phi : C \to \mathbb{CP}^1$ by

$$\phi([x, y, z]) = [x, z].$$

Define the ramification index $\nu_\phi[a, b, c]$ of ϕ at $[a, b, c] \in C$.

(iv) Prove that $\nu_\phi[a, b, c] > 1$ if and only if $[a, b, c] \in C$ and the tangent line to C at $[a, b, c]$ contains the point $[0, 1, 0]$, and that $\nu_\phi[a, b, c] > 2$ if and only if $[a, b, c]$ is a point of inflection of C and the tangent line to C at $[a, b, c]$ contains the point $[0, 1, 0]$.
(You may use that fact that the Hessian $\mathcal{H}_P$ satisfies

$$z^2 \mathcal{H}_P(a, b, c) = (d-1)^2 \det \begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{xy} & P_{yy} & P_y \\ P_x & P_y & dP/(d-1) \end{pmatrix}.)$$

4. (i) The Open Mapping Theorem for Riemann surfaces says that if $f : R \to S$ is a nonconstant holomorphic map between Riemann surfaces R and S and R is connected, then $f(R)$ is an open subset of S.

   Use this result to prove that a nonconstant holomorphic map $f : R \to S$ between connected Riemann surfaces R and S is surjective when R is compact. Deduce that S is also compact in this case.

   (ii) Using part (i) or otherwise, prove that if R is a compact connected Riemann surface then there are no nonconstant holomorphic functions $f : R \to \mathbb{C}$.

   (iii) Let S be a compact connected Riemann surface of genus zero. Assuming that the Riemann-Roch theorem applies to S, show that if p is any point on S and D is the divisor $D = p$, then $l(D) = 2$. State clearly any result you use.

   (iv) Deduce that there exists a meromorphic function f on S with a simple pole at p and no other poles.

   (v) Let f be a meromorphic function on S with the properties given in part (iv). Show that $f : S \to \mathbb{CP}^1$ is a holomorphic bijection. (You may use the fact that a nonconstant holomorphic map $f : R \to S$ between connected compact Riemann surfaces takes each value in S the same number of times counting multiplicity.)

5. (i) Define a *holomorphic atlas* on a surface.

   (ii) Write down a holomorphic atlas on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, verifying that the atlas is holomorphic.

   (iii) Give two definitions of a meromorphic differential on a Riemann surface. (One definition should be in terms of pairs of meromorphic functions. The other should be in terms of a collection of local meromorphic functions associated to a holomorphic atlas). Prove that these two definitions are equivalent. (You may assume that every Riemann surface admits at least one nonconstant meromorphic function.)

   (iv) The holomorphic differential dz on $\mathbb{C}$ extends to a meromorphic differential on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Find the poles of dz viewed as a meromorphic differential on $\mathbb{CP}^1$ and at each pole find its order.