1. (i) Let k be a field and let $0 \neq f \in k[x,y]$ be a polynomial. Define what it means for a point $P \in k^2$ on $f = 0$ to be a *singular point* on $f = 0$. Give an example of a polynomial $0 \neq f \in \mathbb{R}[x,y]$ such that every $P \in \mathbb{R}^2$ on $f = 0$ is non-singular, but such that there exists a singular point $Q \in \mathbb{C}^2$ on $f = 0$.

   (ii) Now let k be an arbitrary field, and set $f = y^2 - x^4 \in k[x,y]$. Prove that $(0,0)$ is always a singular point on $f = 0$. Give an example of a field k for which *all* k-points on $f = 0$ are singular. Is there an example of a field k such that all k-points on the graph $y^2 - x^5 = 0$ are singular?

2. Assuming any results from the course that you may need, find all rational solutions to $x^2 - 2y^2 = 1$.

3. Let p be a prime. We proved in the course that if $\alpha \in \mathbb{Z}_p$ then α has a unique expansion as $\alpha = \sum_{n \geq 0} a_n p^n$ with $a_n \in \{0, 1, 2, \ldots, p-1\}$, and in this question you may assume this result and also any other standard facts we proved about the a_n. Now set $p = 3$, and consider the expansions of the following elements of $\mathbb{Z}_3$ as above.

   (i) If $\alpha = 25 = \sum_{n \geq 0} a_n 3^n$ then what is a_2?

   (ii) If $\alpha = 1/2 = \sum_{n \geq 0} a_n 3^n$ then what is a_2?

   (iii) If $\alpha = -10 = \sum_{n \geq 0} a_n 3^n$ then what is a_{2007}?

   (iv) If $\alpha = 1/8 = \sum_{n \geq 0} a_n 3^n$ then what is a_{2007}?

4. Prove that there are infinitely many rational solutions to $x^3 + y^3 = 9$. You may assume that if $[r : s : t] \in \mathbb{P}^2(\mathbb{Q})$ is a rational point on $F = 0$, where F is the associated homogeneous cubic, then the tangent to $F = 0$ at this point hits the cubic again at $[r(r^3 + 2s^3) : -s(2r^3 + s^3) : t(r^3 - s^3)]$.

5. You may assume any standard theorems from the course in this question.

   (i) *Assuming* that the equation $y^2 = x^3 + 4$ has only finitely many rational solutions, find them all.

   (ii) Prove that the equation $y^2 = x^3 + 3$ has infinitely many rational solutions.