1. (a) Let k be a field, and let $0 \neq f \in k[x,y]$ be a polynomial. What does it mean for a point $P = (a,b) \in k^2$ to be a *singular point* on $f = 0$?

   (b) Now let k be the field $\mathbb{Q}$ of rational numbers, and consider the polynomial

$$f(x,y) = x^2 + 4xy + 3y^2 - 4x - 8y + 4.$$

   (i) Find a singular point on $f = 0$.

   (ii) Hence or otherwise, find all solutions to $f(a,b) = 0$ with $a, b \in \mathbb{Q}$.

2. (a) State and prove Hensel's Lemma.

   (b) Let $\mathbb{Z}_3$ denote the 3-adic integers. For each of the following polynomials $F(X) \in \mathbb{Z}_3[X]$, either prove that there exists some $a \in \mathbb{Z}_3$ such that $F(a) = 0$, or prove that no $a \in \mathbb{Z}_3$ satisfies $F(a) = 0$.

   (i) $F(X) = X^4 - X + 3$.

   (ii) $F(X) = X^4 - X + 2$.

   (iii) $F(X) = X^3 - 12$.

3. (a) What does it mean for a subset V of $\mathbb{R}^n$ to be *convex*? What does it mean for V to be *symmetric*? Prove that if V is non-empty, convex and symmetric, then V contains the origin.

   (b) Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice of finite index m, and let V be a convex and symmetric subset of $\mathbb{R}^n$. State a theorem giving a criterion which guarantees that V contains a non-zero lattice point of Λ.

   (c) Let N denote any positive divisor of $10^{10} + 2$. Using the result in Part (b) above, prove that N can be written as $a^2 + 2b^2$ with a and b integers.

4. Let E denote the elliptic curve $Y^2 + Y = X^3$ over $\mathbb{Q}$. Assuming any standard results from the course, compute the torsion subgroup of $E(\mathbb{Q})$ and prove that it is generated by the point $(0,0)$.

5. Let E denote the elliptic curve $Y^2 = X(X-1)(X+3)$ over $\mathbb{Q}$. Assuming any standard results from the course, prove that $E(\mathbb{Q})$ is finite.