Imperial College
London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M4A9 Statistical Mechanics II

Date: Thursday, 26th May 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The Hamiltonian $\mathcal{H}(p,q)$ of a magnetic system in a magnetic field B is a function of s generalised coordinates $q \equiv (q_1, q_2, \ldots q_s)$, s generalised momenta $p \equiv (p_1, p_2, \ldots p_s)$ and n magnetic moments $m \equiv (m_1, m_2, \ldots m_n)$ and is given by

$$\mathcal{H}(p,q,m) = \mathcal{H}_0(p,q) - B \sum_{i=1}^{n} m_i$$

where m_i can take on the values $\pm m_0$ only.

Write down an expression for the partition function $\mathcal{Z}(\beta, V, B)$ in the canonical ensemble where $\beta = 1/k_B T$ and V is the volume.

Show that the mean values of the total magnetic moment ($M = \sum_{i=1}^{n} m_i$) and the square of the moment are given by

$$\overline{M} = \frac{1}{\beta \mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial B} \quad \text{and} \quad \overline{M^2} = \frac{1}{\beta^2 \mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial B^2}.$$

Show by a general argument that the mean square fluctuation in M is given by

$$\overline{(M - \overline{M})^2} = k_B T \, \chi \qquad (1)$$

where the susceptibility $\chi = \partial \overline{M}/\partial B$.

Calculate $\overline{M}$, $\chi = \partial \overline{M}/\partial B$ and $\overline{M^2}$ explicitly and hence verify result (1) above by direct substitution.

Show that result (1) remains true even when there is an additional term in the Hamiltonian of the form

$$\sum_{ij} J_{ij} \, m_i \, m_j \, .$$

2. (a) Consider the N spin, one dimensional Ising model in zero field

$$E = -J \sum_{i=1}^{N} s_i s_{i+1}$$

with periodic boundary conditions $s_1 = s_{N+1}$.

Use the transfer matrix method to show that in the limit $N \to \infty$, the free energy per site is

$$f = -k_B T \ln 2 \cosh \beta J$$

(b) For the N spin, Ising model with free boundary conditions

$$E = -J \sum_{i=1}^{N-1} s_i s_{i+1}$$

derive the recurrence relation for the partition function Z_N

$$Z_{N+1} = Z_N 2 \cos h\beta J$$

Hence show that in the limit $N \to \infty$, the free -energy per site is the same as the result for the periodic Ising model shown above

3. (a) For a general Ising model

$$E = -J \sum_{<ij>} s_i s_j - H \sum s_i$$

derive the susceptibility relation

$$k_B T \frac{\partial m}{\partial H} = \sum_i G(0, i)$$

Where $G(i, j)$ is the spin-spin correlation function.

(b) Near the critical temperature the correlation function shows scaling behaviour

$$G(r) = r^{-(d-2+\eta)} \Lambda \pm (r t^{\nu}; H t^{-\Delta})$$

where $t \equiv |T_c - T|/T_c$, r is the distance between the spins and $\pm$ refers to $T > T_c$ and $T < T_c$. Show how this scaling leads to the critical exponent relation

$$\gamma = (2 - \eta)\nu$$

4. (a) Prove that

$$\int_{\mathbb{R}^3} Q(f,f)(\ln f)(\vec{\xi})d\xi \leq 0$$

where equality holds if $f(\vec{\xi}) = Ae^{\vec{b}\cdot\vec{\xi}+c\vec{\xi}^2}$ for some $A, \vec{b}, c$ constants. You may use that

$$\int_{\mathbb{R}^3} Q(f,f)\varphi(\vec{\xi})d\vec{\xi} = \int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_0^{2\pi}\int_0^{\pi/2}(f'f_1' - ff_1)(\varphi + \varphi_1 - \varphi' = \varphi_1')B(\theta, V)d\theta d\varepsilon d\vec{\xi}_1 d\vec{\xi}$$

whenever the integral makes sense and that

$$(x-y)\ln\frac{y}{x} \leq 0 \text{ for all } x, y > 0.$$

(b) Write down the definition of a collisional invariant in $\mathbb{R}^2$

Then show that the function

$$g(\xi) = \xi_x\vec{\xi}^2, \quad \vec{\xi} = (\xi_x, \xi_y) \in \mathbb{R}^2$$

is not a collisional invariant.

5. Show that the principal momentum $P_M : (\rho, \rho\vec{v}, \rho(2\epsilon) + \vec{v}^2)$ of a Maxwellian distribution

$$f_M(\vec{\xi}) = ae^{-b(\vec{\xi}-\vec{\lambda})^2}, \quad \vec{\xi} \in \mathbb{R}^3$$

uniquely determines the constants a, b and $\vec{\lambda}$.