**1.** A particle of mass $m$ moving in a time-dependent potential, in one dimension, has a Hamiltonian of the form

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x,t) \ .$$

The potential is abruptly changed at time $t = 0$, so that

$$U(x,t) = \begin{cases} U_i(x), & \text{for} \quad t < 0 \\ U_f(x), & \text{for} \quad t > 0 \ . \end{cases}$$

For $t < 0$ the particle is in the eigen-state $\psi_n^{(i)}(x)$ of the initial Hamiltonian. Write down an expression for the probability that the particle is in the eigen-state $\psi_m^{(f)}(x)$ of the final Hamiltonian for $t > 0$. In particular, obtain the probability $w$ that the particle remains in the ground state.

Determine the normalised ground state wave–function for a particle in the 'box' potential

$$U(x) = \begin{cases} 0, & \text{for} \quad |x| < a/2 \\ +\infty, & \text{for} \quad |x| > a/2 \ . \end{cases}$$

At $t = 0$, the width of the box is suddenly increased from $a_i$ to $a_f > a_i$. Obtain the probability $w$ as a function of the ratio $\nu = a_f/a_i$. Work out the limit

$$\lim_{\nu \to 1} w(\nu)$$

and explain the result.

**2.** For a quantum particle of mass $m$ in a one–dimensional potential well, the WKB quantisation rule is

$$\int_{a_1}^{a_2} p(x)dx = \pi\hbar(n + 1/2),$$

where $p(x)$ is the classical momentum and $a_1$ and $a_2$ are the classical turning points ($n = 0, 1, 2...$).

Within this framework, determine the WKB energy levels, $E_n$, for a particle in the potential of the form

$$U(x) = Ax^6 \ ,$$

where $A$ is a constant. [Express the result in terms of the constant $I = \int_0^1 dt\sqrt{1 - t^6}$ but do not calculate this integral.]

(i) For large $n \gg 1$, the energy distance $\Delta E_n = E_{n+1} - E_n$ between the neighbouring levels scales as $\Delta E_n \sim n^\beta$. Determine $\beta$. Show how the same result can be obtained by using the classical frequency $\omega = 2\pi/T$, where the classical period is given by

$$T = 2m \int_{a_1}^{a_2} \frac{dx}{p(x)} \ .$$

(ii) Next assume that there are $N$ non-interacting electrons (i.e. spin–1/2 particles obeying the Pauli principle) placed into the above potential well. For $N \gg 1$, estimate the Fermi energy and the total ground–state energy of the system.

**3.** (i) Three spin–1/2's interact via the exchange interaction of the strength $J$:

$$\hat{H} = J\,(\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2 + \hat{\mathbf{s}}_2 \cdot \hat{\mathbf{s}}_3 + \hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_3)$$

where $\hat{\mathbf{s}}_i$'s $(i = 1, 2, 3)$ are spin–1/2 operators.

Determine all the eigenvalues of the Hamiltonian $\hat{H}$.

[Hint: relate the exchange Hamiltonian to the total spin operator; total spin for a complex of three spin–1/2's can take values 1/2 or 3/2.]

**(ii)** Four identical Bose particles occupy two different quantum states $\psi_i(\xi)$, $i = 1, 2$. The wave–functions $\psi_i(\xi)$ are normalised and mutually orthogonal. Determine the normalised four–particle wave–functions $\Psi(\xi_1, \xi_2, \xi_3, \xi_4)$ such that the four particles are grouped in two pairs, each pair occupying the same quantum state.

**4.** The Hamiltonian of a three–dimensional harmonic oscillator of mass $m$ and frequency $\omega$ is given by:

$$\hat{H}_0 = \sum_{i=1}^{3} \left[ \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_i^2 \right] ,$$

where $\hat{x}_i$ and $\hat{p}_i$ are the canonical coordinate and momentum operators with the commutation relation $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$ (in units such that $\hbar = 1$).

The annihilation and creation operators are defined by:

$$\hat{a}_i = \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x}_i + i\hat{p}_i), \quad \hat{a}_i^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x}_i - i\hat{p}_i) .$$

Show that the creation and annihilation operators satisfy the commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$$

and derive the second–quantized version of the Hamiltonian of the three–dimensional harmonic oscillator.

An external non-linear perturbation

$$\hat{V} = \alpha \hat{x}_1^2 \hat{x}_2{}^2 \hat{x}_3{}^4$$

is now applied to the oscillator.

Express $\hat{V}$ in terms of the annihilation and creation operators.

Using the commutation relations, find the first-order correction to the ground-state energy due to the perturbation $\hat{V}$.

5. A particle of mass $m$ moves in the periodic 'brush' potential of the form

$$U(x) = \lambda \sum_{i=-\infty}^{+\infty} \delta(x - ia) \, ,$$

where $i$ is an integer, $a$ is the period of the potential, and $\lambda > 0$. The energy bands $\epsilon_n(k)$ $(n = 0, 1, 2, ...)$ are determined from the dispersion relation

$$\cos(ka) = \cos(pa) + \frac{\alpha}{p} \sin(pa) \, ,$$

where $E = \hbar^2 p^2/(2m)$ and $\alpha = m\lambda/\hbar^2$.

Determine the asymptotic form of the energy gaps at the edge of the Brillouin zone, $k = \pi/a$,

$$\Delta\epsilon_n = \epsilon_{n+1}(\pi/a) - \epsilon_n(\pi/a)$$

($n$ even) in the limit of high energies.

[Hint: the gaps required above are between the closest pairs of energy eigenvalues.]