1. Consider the problem of homogenization for the two–point boundary value problem

$$-\frac{d}{dx}\left(a\left(\frac{x}{\epsilon}\right)\frac{du^\epsilon(x)}{dx}\right) = f(x) \ \text{ for } x \in (0,L),$$
$$u^\epsilon(0) = u^\epsilon(L) = 1.$$

The coefficient $a(y)$ is smooth, 1–periodic and satisfies

$$0 < \alpha \leq a(y) \leq \beta,$$

for some positive constants α, β. The function $f(x)$ is also smooth.

(a) Write down the homogenized equation, the formula for the homogenized coefficient and the cell problem.

(b) Solve the cell problem to show that the homogenized coefficient is

$$\overline{a} = \frac{1}{\int_0^1 a(y)^{-1}\, dy}.$$

(c) Show that

$$\alpha \leq \overline{a} \leq \beta$$

and that

$$\overline{a} \leq \int_0^1 a(y)\, dy.$$

2. Consider the initial value problem

$$\begin{aligned}\frac{\partial u^\epsilon}{\partial t} &= \left(b_1(x) + \frac{1}{\epsilon} b_2\left(\frac{x}{\epsilon}\right)\right)\frac{\partial u^\epsilon}{\partial x} + D\frac{\partial^2 u^\epsilon}{\partial x^2} \quad \text{for } (x,t) \in \mathbb{R}\times\mathbb{R}^+, && (1a)\\ u^\epsilon &= f(x) \quad \text{for } (x,t)\in\mathbb{R}\times\{0\}, && (1b)\end{aligned}$$

where D is a positive constant, the function $b_2(y)$ is smooth and 1–periodic in y and $b_1(x)$ is smooth in x. Use the method of multiple scales to homogenize the above PDE. In particular:

(a) Show that a necessary condition in order to be able to homogenize (1) is the centering condition

$$\int_0^1 b_2(y)\rho(y)\,dy = 0 \tag{2}$$

where $\rho(y)$ is the unique solution of

$$-\frac{d}{dy}\left(b_2(y)\rho(y)\right) + D\frac{d^2\rho(y)}{dy^2} = 0, \qquad \int_0^1 \rho(y)\,dy = 1.$$

on $[0,1]$ with periodic boundary conditions.

(b) Assuming that (2) holds, show that the homogenized equation is

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial u}{\partial x} + \mathcal{K}\frac{\partial^2 u}{\partial x^2}.$$

(c) Show that the formulas for the homogenized coefficients are

$$b(x) = b_1(x) + \int_0^1 \frac{d\chi}{dy}(y)\rho(y)\,dy$$

and

$$\mathcal{K} = \int_0^1 \left(b_2(y)\chi(y) + 2D\frac{d\chi}{dy}(y) + D\right)\rho(y)\,dy$$

where $\chi(y)$ is the solution of

$$-b_2(y)\frac{d\chi}{dy} - D\frac{d^2\chi}{dy^2} = b_2(y)$$

on $[0,1]$ with periodic boundary conditions.

3. Let $V(y)$ be a smooth 1–periodic function, D a positive constant and consider the differential operator

$$\mathcal{L} = -\nabla_y V(y) \bullet \nabla_y + D\Delta_y$$

on $\mathcal{Y} = [0,1]^d$, equipped with periodic boundary conditions. Let $\mathcal{L}^*$ denote the L^2–adjoint of $\mathcal{L}$.

(a) Show that the Gibbs distribution

$$\rho(y) = \frac{1}{Z} e^{-V(y)/D}, \quad Z = \int_{\mathcal{Y}} e^{-V(y)/D}\, dy$$

is a solution of the equation

$$\mathcal{L}^* \rho = 0, \quad \int_{\mathcal{Y}} \rho(y)\, dy = 1.$$

on $\mathcal{Y}$ with periodic boundary conditions.

(b) Let $b(y) = -\nabla_y V(y)$. Show that

$$\int_{\mathcal{Y}} b(y)\rho(y)\, dy = 0.$$

(c) Show that

$$\int_{\mathcal{Y}} f(y)\mathcal{L}h(y)\rho(y)\, dy = \int_{\mathcal{Y}} (\mathcal{L}f(y))\, h(y)\rho(y)\, dy$$

for all $f,\, h \in C^2_{per}(\mathcal{Y})$.

4. Consider the stochastic differential equation (SDE)

$$dy = -\alpha y \, dt + \sqrt{2\lambda} \, dW$$

where $W(t)$ is a standard one-dimensional Brownian motion and $\alpha, \ \lambda$ are positive constants.

(a) Write down the generator and the forward and backward Kolmogorov equations corresponding to this SDE.

(b) Show that the solution of this SDE is

$$y(t) = e^{-\alpha t} y(0) + \sqrt{2\lambda} \int_0^t e^{-\alpha(t-s)} \, dW(s).$$

(c) Assume that the initial condition is non-random. Show that

$$\mathbb{E} y(t) = e^{-\alpha t} y(0)$$

and

$$\mathbb{E}\big(y(t) - \mathbb{E} y(t)\big)^2 = \frac{\lambda}{\alpha} \left(1 - e^{-2\alpha t}\right).$$

5. Consider the system of SDEs

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{\epsilon}(1-y^2)x, && (3a)\\
\frac{dy}{dt} &= -\frac{1}{\epsilon^2}y + \sqrt{\frac{2}{\epsilon^2}}\frac{dW}{dt}, && (3b)
\end{aligned}
$$

where $W(t)$ is a standard one–dimensional Brownian motion. Use the method of multiple scales to obtain the homogenized equation. In particular:

(a) Write down the backward Kolmogorov equation corresponding to (3).

(b) Look for a solution of the Kolmogorov equation in the form of a power series expansion in ϵ and obtain a sequence of equations for the first three terms in the expansion.

(c) Analyze these three equations to obtain the homogenized Kolmogorov equation. Deduce from this that the homogenized SDE is

$$\frac{dX}{dt} = X + \sqrt{2}X\frac{dW}{dt}.$$

You may use without proof the formulas

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}y^2\right)\,dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}y^2\exp\left(-\frac{1}{2}y^2\right)\,dy = 1$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}y^4\exp\left(-\frac{1}{2}y^2\right)\,dy = 3$$