Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M4A39

Homogenisation Theory for PDEs

Date: Thursday, 25th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider the problem of homogenization for

$$\frac{\partial u^\epsilon}{\partial t} = \frac{1}{\epsilon} b\left(\frac{x}{\epsilon}\right)\frac{\partial u^\epsilon}{\partial x} + D\frac{\partial^2 u^\epsilon}{\partial x^2}, \quad (x,\, t) \in \mathbb{R}\times\mathbb{R}^+,$$

$$u^\epsilon(x,0) = u_{in}(x), \quad x \in \mathbb{R},$$

where $b(y) = -\frac{dV}{dy}(y)$, $V(y)$ is 1–periodic, $u_{in}(x)$ is a smooth function and D is constant.

(a) Write down the homogenized equation, the formula for the effective diffusion coefficient and the cell problem.

(b) Solve the cell problem to show that the effective diffusion coefficient is given by the formula

$$\overline{D} = \frac{D}{\left(\int_0^1 e^{-V(y)/D}\, dy\right)\left(\int_0^1 e^{V(y)/D}\, dy\right)}.$$

(c) Calculate $\overline{D}$ for the case

$$V(y) = \begin{cases} y & : \quad y \in [0, \frac{1}{2}], \\ 1-y & : \quad y \in (\frac{1}{2}, 1]. \end{cases}$$

2. Consider the problem of homogenization for

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\epsilon}\right)\frac{\partial u^\epsilon}{\partial x_j}\right) = f, \quad \text{for } x \in \Omega,$$

$$u^\epsilon(x) = 0, \quad \text{for } x \in \partial\Omega,$$

where $\{a_{ij}(y)\}_{i,j=1}^{d}$ is smooth, 1–periodic and f is smooth.

(a) Write down the homogenized equation, the cell problem and the formulae for the homogenized coefficients.

(b) Consider the case where $d = 2$ and

$$a_{ij}(y) = a(y)\delta_{ij}, \quad 1\text{–periodic}, \quad i, j = 1, 2,$$

where δ_{ij} stands for Kronecker's delta. Show that the cell problem and the formula for the homogenized coefficients become

$$-\sum_{i=1}^{2} \frac{\partial}{\partial y_i}\left(a(y)\frac{\partial \chi^k}{\partial y_i}\right) = \frac{\partial a(y)}{\partial y_k}, \quad \chi^k(y) \text{ is } 1\text{–periodic}, \ k = 1, 2$$

and

$$\overline{a}_{ij} = \int_Y \left(a(y)\delta_{ij} + a(y)\frac{\partial \chi^j(y)}{\partial y_i}\right) dy, \quad i, j = 1, 2.$$

(c) Assume that $a(y)$ is of the form

$$a(y) = a_1(y_1)a_2(y_2),$$

where both $a_i(y_i)$, $i = 1, 2$ are strictly positive, smooth, 1–periodic functions. Solve the two components of the cell problem by looking for solutions of the form $\chi^k = \chi^k(y_k)$, $k = 1, 2$. Show that the homogenized coefficients are

$$\overline{a}_{11} = \frac{\int_0^1 a_2(y_2)\,dy_2}{\int_0^1 (a_1(y_1))^{-1}\,dy_1}, \quad \overline{a}_{22} = \frac{\int_0^1 a_1(y_1)\,dy_1}{\int_0^1 (a_2(y_2))^{-1}\,dy_2}$$

and $\overline{a}_{ij} = 0$, $i \neq j$.

3. (a) Define

$$u(x) = \begin{cases} 3x^2 & : \text{ for } 0 \leq x \leq \frac{1}{2}, \\ 1 - x^2 & : \text{ for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Show that the weak derivative of $u(x)$ is

$$\frac{du}{dx} = \begin{cases} 6x & : \text{ for } 0 \leq x \leq \frac{1}{2}, \\ -2x & : \text{ for } \frac{1}{2} < x \leq 1. \end{cases}$$

(b) Calculate $\|u\|_{L^2(0,1)}$ and $\left\|\frac{du}{dx}\right\|_{L^2(0,1)}$. Show that $u(x) \in H_0^1(0,1)$.

(c) Let $f(x) \in C_0^\infty(0,1)$. Use the identity

$$\int_0^1 \frac{d}{dx}\left(f(x)^2 x\right)\, dx = 0$$

to prove that $f(x)$ satisfies Poincaré inequality.

4. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and consider the boundary value problem

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\epsilon}\right)\frac{\partial u^\epsilon}{\partial x_j}\right) + \mathbf{b}\left(\frac{x}{\epsilon}\right) \bullet \nabla u^\epsilon = f, \quad x \in \Omega,$$

$$u^\epsilon = 0, \quad x \in \partial\Omega,$$

where $D > 0$ is a constant, $\{a_{ij}(y)\}_{i,j=1}^{d}$, $\mathbf{b}(y)$ are smooth, 1–periodic and $f(x)$ is smooth. Use the method of multiple scales to homogenize the above PDE. In particular:

(a) Show that the homogenized equation is

$$-\sum_{i,j=1}^{d} \overline{a}_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \overline{\mathbf{b}} \bullet \nabla u = f,$$

together with the boundary condition $u = 0$ on $\partial\Omega$.

(b) Show that the formulas for the homogenized coefficients $\overline{\mathbf{b}}$, $\{\overline{a}\}_{ij=1}^{d}$ are given by

$$\begin{aligned} \overline{b}_j &= \int_Y \left(b_j(y) + \sum_{k=1}^{d} b_k(y)\frac{\partial \chi^k}{\partial y_k}(y)\right) dy, \quad j = 1, \ldots, d, \\ \overline{a}_{ij} &= \int_Y \left(a_{ij}(y) + \sum_{k=1}^{d} a_{ik}(y)\frac{\partial \chi^j(y)}{\partial y_k(y)}\right) dy, \quad i,j = 1, \ldots, d. \end{aligned}$$

(c) Show that the cell problem is

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial \chi^k}{\partial y_j}\right) = \sum_{i=1}^{d} \frac{\partial a_{ik}}{\partial y_i}, \quad \chi^j(y) \text{ is 1–periodic, } k = 1, \ldots, d.$$

5. In class we studied the problem of homogenization for the advection–diffusion equation

$$\frac{\partial u^\epsilon(x,t)}{\partial t} + \frac{1}{\epsilon}\mathbf{a}\left(\frac{x}{\epsilon}\right)\bullet\nabla u^\epsilon(x,t) - D\Delta u^\epsilon(x,t) = 0, \quad (x,\, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

$$u^\epsilon(x,0) = u_{in}(x) \quad x \in \mathbb{R}^d.$$

where $\mathbf{a}(y)$ is a 1–periodic, divergence–free vector field with zero average, $\int_Y \mathbf{a}(y)\,dy = 0$. We obtained the homogenized equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} \mathcal{K}_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j},$$

together with $u(x,0) = u_{in}(x)$. The effective diffusivity is given by

$$\mathcal{K}_{ij} := D\delta_{ij} - \frac{1}{2}\int_Y \left(a_i(y)\chi^j(y) + a_j(y)\chi^i(y)\right)\,dy, \quad i,j = 1,\ldots,d,$$

where $Y = [0,1]^d$ and $\chi^j(y)$, $j = 1,\ldots,d$ solves the cell problem

$$-D\Delta_y\chi^j(y) + \mathbf{a}(y)\bullet\nabla_y\chi^j(y) = -a_j(y), \quad j = 1,\ldots,d,$$

with periodic boundary conditions and

$$\int_Y \chi^j(y)\,dy = 0,\ j = 1,\ldots,d.$$

(a) Show that the solution of the cell problem is unique.

(b) Show that the effective diffusivity is given by the formula

$$\mathcal{K}_{ij} := D\left(\delta_{ij} + \int_Y \nabla_y\chi^i(y)\bullet\nabla_y\chi^j(y)\,dy\right), \quad i,j = 1,\ldots,d.$$

(c) Show that the effective diffusivity is positive definite.