1. Consider the problem of homogenization for the following two–point boundary value problem

$$-\frac{d}{dx}\left(a\left(\frac{x}{\epsilon}\right)\frac{du^\epsilon(x)}{dx}\right) = f(x), \quad \text{for } x \in [0, L],$$
$$u^\epsilon(0) = u^\epsilon(L).$$

The coefficient $a(y)$ is smooth, 1–periodic and satisfies

$$0 < \alpha \leq a(y) \leq \beta,$$

for some positive constants α, β. The function $f(x)$ is also smooth.

(a) Write down the homogenized equation, the formula for the homogenized coefficient and the cell problem.

(b) Solve the cell problem to show that

$$\overline{a} = \frac{1}{\int_0^1 a(y)^{-1}\,dy}.$$

(c) Calculate $\overline{a}$ for the case

$$a(y) = \begin{cases} a_1 & : \quad y \in [0, \frac{1}{2}], \\ a_2 & : \quad y \in (\frac{1}{2}, 1], \end{cases}$$

where a_1, a_2 are positive constants.

2. Consider the following initial value problem:

$$\frac{\partial u^\epsilon(x,t)}{\partial t} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\epsilon},\frac{t}{\epsilon^2}\right)\frac{\partial u^\epsilon(x,t)}{\partial x_j}\right) = 0, \quad \text{for } (x,t) \in \mathbb{R}^d \times (0,T), \tag{2a}$$

$$u^\epsilon(x,0) = u_{in}(x) \ \text{ in } \mathbb{R}^d. \tag{2b}$$

with $A(y,\tau) = \{a_{ij}(y,\tau)\}_{i,j=1}^d$ smooth, bounded, periodic with period 1 in y_i, $i = 1, \dots d$ and τ. The initial conditions $u_{in}(x)$ are also smooth.

Use the method of multiple scales to show that

(a) The cell problem is

$$\left[\frac{\partial}{\partial \tau} - \sum_{i,j=1}^{d} \frac{\partial}{\partial y_i}\left(a_{ij}(y,\tau)\frac{\partial}{\partial y_j}\right)\right]\theta^\ell(y,\tau) = \sum_{i=1}^{d} \frac{\partial a_{i\ell}(y,\tau)}{\partial y_i}, \quad \ell = 1,\dots,d,$$

$$\theta^\ell(y,\tau) \ \text{ is } 1\text{–periodic in } y,\ \tau, \quad \int_0^1 \int_Y \theta^\ell(y,\tau)\, dy d\tau = 0,$$

where $Y = [0,1]^d$.

(b) The homogenized equation is

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} \overline{a}_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

(c) The homogenized coefficients are given by the formula

$$\overline{a}_{ij} = \int_0^1 \int_Y \left(a_{ij}(y,\tau) + a_{ik}(y,\tau)\frac{\partial \theta^j(y,\tau)}{\partial y_k}\right) dy d\tau, \quad i,j = 1,\dots,d.$$

3. Let Ω be a bounded subset of $\mathbb{R}^d$ with smooth boundary.

(a) Give the definition of the first weak derivative of $u(x) \in L^1(\Omega)$ with respect to x_i, $i = 1, \ldots, d$. Give the definition of the Sobolev space $H^1(\Omega)$. Write down the inner product and norm of $H^1(\Omega)$.

(b) Let $\Omega = (0,1) \in \mathbb{R}$ and let $f_1(x) = 1 - x$, $f_2(x) = x^2$. Calculate $(f_1, f_2)_{H^1(0,1)}$, $\|f_1\|_{H^1(0,1)}$ and $\|f_2\|_{H^1(0,1)}$.

(c) Let $\Omega = (0,1) \in \mathbb{R}$. Define

$$u(x) = \begin{cases} x & : \text{ for } 0 \leq x \leq \frac{1}{2}, \\ 1 - x & : \text{ for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Show that the weak derivative of $u(x)$ is

$$\frac{du}{dx} = \begin{cases} 1 & : \text{ for } 0 \leq x \leq \frac{1}{2}, \\ -1 & : \text{ for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

4. In class we studied the problem of homogenization for the advection–diffusion equation

$$\frac{\partial u^\epsilon(x,t)}{\partial t} + \frac{1}{\epsilon}\mathbf{a}\left(\frac{x}{\epsilon}\right)\bullet\nabla u^\epsilon(x,t) - \kappa\Delta u^\epsilon(x,t) = 0, \text{ for } (x,t) \in \mathbb{R}^d \times (0,T),$$

$$u^\epsilon(x,0) = u_{in}(x) \text{ in } \mathbb{R}^d.$$

where $\mathbf{a}(y)$ is a 1–periodic, divergence–free vector field with zero average, $\int_Y \mathbf{a}(y)\,dy = 0$. We obtained the homogenized equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^d \mathcal{K}_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j},$$

together with $u(x,0) = u_{in}(x)$. The effective diffusivity is given by

$$\mathcal{K}_{ij} := \kappa\delta_{ij} - \int_Y a_i(y)\chi^j(y)\,dy, \quad i,j = 1,\dots d,$$

where $Y = [0,1]^d$ and $\chi^j(y)$, $j = 1,\dots,d$ solves the cell problem

$$-\kappa\Delta_y\chi^j(y) + \mathbf{a}(y)\bullet\nabla_y\chi^j(y) = -a_j(y), \quad j = 1,\dots,d,$$

with periodic boundary conditions and

$$\int_Y \chi^j(y)\,dy = 0,\ j = 1,\dots d.$$

Consider now the two–dimensional velocity field

$$\mathbf{a}(y) = (0, sin(2\pi y_1)). \tag{3}$$

(a) Write down the two components of the cell problem. Show that $\chi^1(y) = 0$.

(b) Compute $\chi^2(y)$.

(c) Compute the effective diffusivity.

5. We let Ω denote a bounded subset of $\mathbb{R}^d$ with smooth boundary and Y denote the unit cube in $\mathbb{R}^d$, $Y := [0,1]^d$.

(a) Give the definition of two–scale convergence .

(b) Let $u^\epsilon(x)$ be a sequence in $L^2(\Omega)$ which two–scale converges to $u_0(x,y) \in L^2(\Omega \times Y)$. Show that u^ϵ converges weakly in $L^2(\Omega)$ to

$$\overline{u}_0(x) := \int_Y u_0(x,y)\,dy.$$

(c) Let $u^\epsilon(x)$ be a uniformly bounded sequence in $H^1(\Omega)$. Prove that $u^\epsilon(x)$ two–scale converges to the weak–H^1 limit of $u^\epsilon(x)$.