1. Let $f : X \to X$ be a continuous map of a compact metric space. Let $\mathcal{M}$ denote the space of all Borel probability measures on X.

(i) (a) Define the weak-star topology on $\mathcal{M}$ by giving the definition of a converging sequence of measure in $\mathcal{M}$.

(b) Define the *pullback* map $f_* : \mathcal{M} \to \mathcal{M}$ induced by the map f.

(c) Show that f_* is continuous in the weak-star topology (you may assume the fact that $\int \varphi d(f_*\mu) = \int \varphi \circ f d\mu$).

(ii) Let

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} f_*^i \delta_x.$$

(a) Say which property of the weak-star topology guarantees that the sequence $\{\mu_n\}$ always has a limit point.

(b) Show that any limit point μ of the sequence $\{\mu_n\}$ is f-invariant.

(iii) Give an example of a continuous function on a non-compact set X which admits no invariant measures. Justify your answer.

(iv) Let $X = \Sigma_2^+$ and $f = \sigma$ the shift map. Give an example of a point x for which the corresponding sequence μ_n has more than one limit point.

2. Let $f : I \to I$ be an interval map and μ a probability measure.

(i) (a) Say what it means for μ to be invariant, ergodic, absolutely continuous with respect to Lebesgue measure.

(b) State Birkhoff's ergodic theorem.

(ii) (a) Let $k \in \mathbb{N}$, $k \geq 2$. Say what it means for a number x to be normal in base k.

(b) Apply Birkhoff's ergodic theorem to show that Lebesgue almost every real number is normal in *every* base k.

(iii) (a) Say what is means for an invariant measure to be mixing.

(b) Given two functions $\varphi, \psi : I \to \mathbb{R}$, define the correlation function $C_n(\varphi, \psi)$.

(c) Explain how the decay of the correlation function is related to the notion of mixing.

(d) Show that mixing implies ergodicity

(e) Give an example of a map with an invariabnt probability measure which is absolutely continuous with respect to Lebesgue and which is ergodic but not mixing. Justify your answer.

3. Let $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ denote the space of all infinite sequences of 0 and 1 with the standard metric.

(i) (a) Define the cylinder sets $I_{a_1 \dots a_n}$.

(b) For $p \in (0,1)$ define the Markov measure μ_p on cylinder sets.

(ii) Show that μ_p is invariant under the shift map σ.

(iii) Show that there exists a family $\{\mathcal{A}_p\}_{p\in(0,1)}$ of *disjoint sets* in Σ_2^+ such that for each $p \in (0,1)$

$$\mu_p(\mathcal{A}_p) = 1.$$

You may assume if necessary that μ_p is ergodic for every $p \in (0,1)$.

4. Let $f : I \to I$ be an interval map.

(i) (a) Define the *Markov* property of f.

(b) Define the notion of *cylinder sets* for a Markov map f.

(c) Define the *bounded distortion* property of a Markov map.

(ii) Suppose that f is a Markov map with the bounded distortion property and assume that the size of cylinder sets of order n tends to 0 as n tends to infinity.
Show that f is ergodic with respect to Lebesgue measure.

(iii) Give an example of a Markov map for which some cylinder sets of order n do not tend to 0 as n tends to infinity.

5. (i) (a) Say what it means for f to admit an *induced Markov map F.*

(b) Say what it means for F to have *integrable return times.*

(c) Suppose that ν is an F-invariant probability measure. Define an f-invariant probability measure $\widehat{\mu}$ in terms of ν .

(d) Show that the integrability of the return times imples that $\widehat{\mu}$ can be normalized to a probability measure and state this normalization explicitly.

(e) Suppose ν is absolutely continuous with respect to Lebesgue measure. Show that μ is also absolutely continuous with respect to Lebesgue measure.

(ii) Let $I = (0,1)$ and let $\mathcal{P} = \{I_0, I_1, I_2\}$ be a partition with $I_0 = (0,1/4)$, $I_1 = (1/4,1/2)$ and $I_2 = (1/2,1)$. Let $f : I \to I$ be a piecewise affine (constant derivative on each partition element) map, mapping I_0 to $(0,1/2) = I_0 \cup I_1$ bijectively and I_1 and I_2 to I bijectively. Show that f admits and induced Markov map with integrable return times by inducing on $\Delta = I_0$.