Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M4A36

Probabilistic Methods in Dynamics

Date: Friday 26th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $D \subset \mathbb{R}^2$ be the closed unit disc and let $f : D \to D$ be a continuous function.

   (a) Give the definition of the omega-limit $\omega(x_0)$ of a point x_0.

   (b) Show that $\omega(x_0)$ is

      (i) nonempty,

      (ii) invariant,

      (iii) closed.

   (c) (i) Write what it means for f to be a contraction.

      (ii) State, without details, the steps of the argument required to show that a contraction has a unique fixed point.

   (d) State without proof the Bendixson-Poincaré theorem.

2. (a) Let $f : X \to X$ be a map on a metric space X. Let $x_0 \in X$. Write what it means to say that the orbit of x_0 is

      (i) periodic,

      (ii) dense.

   (b) Prove that a for a rigid circle rotation

      (i) all orbits are periodic if and only if ρ is rational,

      (ii) all orbits are dense if and only if ρ is irrational.

3. Let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be circle homeomorphisms.

   (a) Write what is means for f, g to be

      (i) topologically *semi-conjugate*,

      (ii) topologically *conjugate*.

   (b) Define the rotation number $\rho(f)$.

   (c) Suppose that $\rho(f)$ is irrational. What can you say about the topological structure of the dynamics of f? Cite any standard results which you need.

   (d) Suppose that $\rho(f) = \rho(g)$ is irrational. What additional conditions guarantee that they are topologically conjugate? Cite any standard results that you need.

   (e) Construct an example of two circle homeomorphisms f and g with $\rho(f) = \rho(g)$ irrational but which are not topologically conjugate.

4. (a) Let $\mathcal{F}$ be a family of analytic maps from a domain $\mathcal{D}$ to the Riemann sphere $\bar{\mathbb{C}}$.

   (i) Write what it means for the family $\mathcal{F}$ to be a *normal family*.

   (ii) Show that if $\mathcal{F}$ is uniformly bounded on each compact set $K \subset \mathcal{D}$ then it is normal. Cite any theorems that you use.

   (iii) State Montel's theorem on normal families.

   (b) Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map. Define

   (i) the Fatou set F_c,

   (ii) the Julia set J_c.

   (c) Let $f_c : \mathbb{C} \to \mathbb{C}$ be the family of polynomial maps $f_c(z) = z^2 + c$.

   (i) Define the Mandelbrot set $\mathcal{M}$.

   (ii) Give an example of Julia set, for a particular parameter value, to show that $\mathcal{M} \neq \emptyset$.

5. (a) Define a Borel probability measure μ on $[1, 0]$.

   (b) Write what it means for μ to be invariant.

   (c) Write what it means for μ to be ergodic.

   (d) State Birkhoff's Ergodic Theorem.

   (e) Every real number in $[0, 1]$ can be written in base k for $k \in \mathbb{N}$, $k \geq 2$, by writing

$$x = \sum_{n=1}^{\infty} \frac{a_n}{k^n}$$

   for some coefficients $a_n \in \{0, \dots, k-1\}$.

   A number x is *normal* in base k if the asymptotic frequency of the occurrence of each digit $a_0, \dots, a_{k-1}$ in the expansion in base k of x is exactly $1/k$. Apply Birkhoff's ergodic theorem to prove that almost all real numbers are normal in every integer base. Cite any standard results that you use.