Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

and MSc EXAMINATIONS

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M4A35/MSA5 Bifurcation Theory

Date: Wednesday, 25th May 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Suppose $\mathbf{F} : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^N$ is a smooth function and $(\mathbf{x}^\star, \lambda^\star) \in \mathbb{R}^N \times \mathbb{R}$ satisfies

$$\mathbf{F}(\mathbf{x}^\star, \lambda^\star) = \mathbf{0}.$$

Let $\mathsf{J}(\mathbf{x}^\star, \lambda^\star) \in \mathbb{R}^{N\times N}$ denote the Jacobian matrix of $\mathbf{F}$ at $(\mathbf{x}^\star, \lambda^\star)$ and $\mathbf{F}_\lambda(\mathbf{x}^\star, \lambda^\star) \in \mathbb{R}^N$ denote the partial derivative of $\mathbf{F}$ with respect to λ at $(\mathbf{x}^\star, \lambda^\star)$.

(a) If $\mathsf{J}(\mathbf{x}^\star, \lambda^\star)$ is non-singular, state carefully the conclusion of the Implicit Function Theorem applied to $\mathbf{F}$ at $(\mathbf{x}^\star, \lambda^\star)$. Explain clearly how to use continuation with respect to λ to compute any zero of $\mathbf{F}$ near $(\mathbf{x}^\star, \lambda^\star)$. Your explanation should include the Newton iteration employed and also how $\mathbf{F}_\lambda(\mathbf{x}^\star, \lambda^\star)$ is used to obtain an accurate starting value for it.

(b) If

$$[\mathsf{J}(\mathbf{x}^\star, \lambda^\star)\ \mathbf{F}_\lambda(\mathbf{x}^\star, \lambda^\star)] \in \mathbb{R}^{N\times(N+1)}$$

has rank N, explain clearly how to use generalised continuation to compute zeroes of $\mathbf{F}$ near $(\mathbf{x}^\star, \lambda^\star)$. Your explanation should include how to choose $(\mathbf{y}^\star, \mu^\star) \in \mathbb{R}^N \times \mathbb{R}$ so that the conditions of the Implicit Function Theorem apply to the function $\mathbf{G} : (\mathbb{R}^N \times \mathbb{R}) \times \mathbb{R} \mapsto \mathbb{R}^N \times \mathbb{R}$ defined by

$$\mathbf{G}(\mathbf{x}, \lambda; \varepsilon) \equiv \left\{ \begin{array}{c} \mathbf{F}(\mathbf{x}, \lambda) \\ \mathbf{y}^\star \cdot [\mathbf{x} - \mathbf{x}^\star] + \mu^\star[\lambda - \lambda^\star] - \varepsilon \end{array} \right\}$$

at the point $(\mathbf{x}^\star, \lambda^\star; 0)$, and what the conclusions of this theorem are. If $(\mathbf{y}^\star, \mu^\star) \in \mathbb{R}^N \times \mathbb{R}$ is normalised so that

$$\mathbf{y}^\star \cdot \mathbf{y}^\star + [\mu^\star]^2 = 1,$$

you should also explain how to obtain an accurate starting value for Newton's method applied to

$$\mathbf{G}(\mathbf{x}, \lambda; \varepsilon) = \mathbf{0}$$

for fixed small $|\varepsilon| \neq 0$.

2. (a) Suppose that $\mathsf{A} \in \mathbb{R}^{N\times N}$ has eigenvalues with strictly negative real part, and also assume that A has N linearly independent eigenvectors. Describe carefully how to define a special inner–product $\langle\, .\,,\, .\,\rangle_\star$ on $\mathbb{R}^N$ so that

$$\langle \mathsf{A}\mathbf{x}, \mathbf{x}\rangle_\star \leq \lambda_{\max} \langle \mathbf{x}, \mathbf{x}\rangle_\star \qquad \forall \mathbf{x} \in \mathbb{R}^N;$$

where $\lambda_{\max} < 0$ is defined by

$$\lambda_{\max} \equiv \max\left\{\mathrm{Re}(\lambda) : \lambda \text{ an eigenvalue of } \mathsf{A}\right\}.$$

(b) Let $\mathbf{x}^\star \in \mathbb{R}^N$ be a stationary solution for the smooth autonomous system

$$(\dagger) \quad \dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \qquad \mathbf{F} : \mathbb{R}^N \mapsto \mathbb{R}^N,$$

i.e. $\mathbf{F}(\mathbf{x}^\star) = \mathbf{0}$. Assume that $\mathsf{J}(\mathbf{x}^\star)$, where $\mathsf{J}(\mathbf{x}^\star)$ is the Jacobian matrix of $\mathbf{F}$ at $\mathbf{x}^\star$, has eigenvalues with strictly negative real part and N linearly independent eigenvectors. Use the special vector norm in (a) to prove that any solution $\mathbf{x}(t)$ of ($\dagger$), with starting value $\mathbf{x}(0)$ sufficiently close to $\mathbf{x}^\star$, will satisfy

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^\star.$$

3. Consider the differential equation

$$(\dagger) \quad \dot{\mathbf{u}}(t) - \mathsf{A}\mathbf{u}(t) = \mathbf{f}(t),$$

where $\mathsf{A} \in \mathbb{R}^{N\times N}$, $\mathbf{f} : \mathbb{R} \mapsto \mathbb{R}^N$ is a continuous function, and $\mathbf{u} : \mathbb{R} \mapsto \mathbb{R}^N$ is our unknown function. State the variation–of–constants formula expressing $\mathbf{u}(t)$ (the solution of ($\dagger$) at time t) in terms of the initial value $\mathbf{u}(0)$, the matrix exponential of A, and the right–hand side $\mathbf{f}$.

(a) Suppose the eigenvalues of A have strictly negative real part, i.e.

$$0 > -\alpha > \max\{\mathrm{Re}(\lambda) : \lambda \text{ an eigenvalue of A}\}$$

for some $\alpha > 0$. Given that for each vector norm (and induced matrix norm) there exists a constant $C \geq 1$ such that

$$\|\mathrm{e}^{\mathsf{A}t}\| \leq C\mathrm{e}^{-\alpha t} \qquad \forall t \geq 0$$

and also assuming that $\mathbf{f}$ satisfies the bound

$$\|\mathbf{f}(t)\| \leq \mathrm{e}^{-2\alpha t} \qquad \forall t \geq 0;$$

deduce that the solution of ($\dagger$) with initial condition $\mathbf{u}(0) = \boldsymbol{\xi}$ satisfies

$$\|\mathbf{u}(t)\| \leq C\mathrm{e}^{-\alpha t}\left\{\|\boldsymbol{\xi}\| + \frac{\mathrm{e}^{-\alpha t}}{\alpha}\right\} \qquad \forall t \geq 0.$$

(b) Suppose the eigenvalues of A have strictly positive real part, i.e.

$$0 < \beta < \min\{\mathrm{Re}(\lambda) : \lambda \text{ an eigenvalue of A}\}$$

for some $\beta > 0$. Given that for each vector norm (and induced matrix norm) there exists a constant $C \geq 1$ such that

$$\|\mathrm{e}^{-\mathsf{A}t}\| \leq C\mathrm{e}^{-\beta t} \qquad \forall t \geq 0$$

and also assuming that $\mathbf{f}$ satisfies the bound

$$\|\mathbf{f}(t)\| \leq \mathrm{e}^{-\gamma t} \qquad \forall t \geq 0$$

for some $\gamma > 0$; explain carefully why there is exactly one initial condition $\mathbf{u}(0)$ so that the solution of ($\dagger$) satisfies

$$\lim_{t\to\infty} \mathbf{u}(t) = \mathbf{0}.$$

Deduce that this unique solution satisfies the bound

$$\|\mathbf{u}(t)\| \leq C\frac{\mathrm{e}^{-\gamma t}}{\gamma} \qquad \forall t \geq 0.$$

4. (a) Suppose $\mathsf{A}(\lambda)$ is an $N \times N$ matrix depending smoothly on a scalar parameter λ. We assume that $\mathsf{A}(\lambda^\star)$ is singular, with one-dimensional right and left null-spaces $\{\boldsymbol{\varphi}^\star\}$ and $\{\boldsymbol{\psi}^\star\}$ respectively, and that this zero eigenvalue of $\mathsf{A}(\lambda^\star)$ is simple, i.e. we can choose the normalisation

$$\boldsymbol{\varphi}^\star \cdot \boldsymbol{\varphi}^\star = 1 \quad \text{and} \quad \boldsymbol{\psi}^\star \cdot \boldsymbol{\varphi}^\star = 1.$$

Explain why the Implicit Function Theorem may be applied at $\lambda = \lambda^\star$ to the augmented nonlinear system

$$(\dagger) \qquad \begin{aligned} \mathsf{A}(\lambda)\mathbf{x} - \mu\mathbf{x} &= \mathbf{0} \\ \boldsymbol{\psi}^\star \cdot \mathbf{x} - 1 &= 0 \end{aligned}$$

for an eigenvalue μ and normalised eigenvector $\mathbf{x}$ of $\mathsf{A}(\lambda)$. If we denote this eigenvalue by $\mu^\star(\lambda)$, so that $\mu^\star(\lambda^\star) = 0$, prove that

$$\frac{\mathrm{d}\mu^\star}{\mathrm{d}\lambda}(\lambda^\star) = \boldsymbol{\psi}^\star \cdot \mathsf{A}'(\lambda^\star)\boldsymbol{\varphi}^\star.$$

(b) Suppose $\mathbf{F} : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^N$ is a smooth function and $\mathsf{J} : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^{N\times N}$ denotes its Jacobian matrix. Suppose $\mathbf{F}$ has the additional property that

$$\mathbf{F}(\mathbf{0}, \lambda) = \mathbf{0} \quad \forall \lambda \in \mathbb{R},$$

and let $\lambda^\star \in \mathbb{R}$ satisfy

- $\operatorname{rank}\{\mathsf{J}(\mathbf{0}, \lambda^\star)\} = N - 1$, with $\{\boldsymbol{\varphi}^\star\}$ denoting the null-space of $\mathsf{J}(\mathbf{0}, \lambda^\star)$ and $\{\boldsymbol{\psi}^\star\}$ denoting the null-space of $\mathsf{J}(\mathbf{0}, \lambda^\star)^T$;
- $\boldsymbol{\psi}^\star \cdot \mathsf{J}_\lambda(\mathbf{0}, \lambda^\star)\boldsymbol{\varphi}^\star \neq 0$, where $\mathsf{J}_\lambda(\mathbf{0}, \lambda^\star) \in \mathbb{R}^{N\times N}$ is the derivative of $\mathsf{J}(\mathbf{0}, \lambda)$ with respect to λ evaluated at $\lambda^\star$.

Explain carefully how the Implicit Function Theorem can be applied to the function $\mathbf{G} : (\{\boldsymbol{\varphi}^\star\}^\perp \times \mathbb{R}) \times \mathbb{R} \mapsto \mathbb{R}^N$, defined by

$$\mathbf{G}(\mathbf{w}, \lambda; \varepsilon) \equiv \begin{cases} \dfrac{1}{\varepsilon}\mathbf{F}(\varepsilon[\boldsymbol{\varphi}^\star + \mathbf{w}], \lambda) & \varepsilon \neq 0 \\ \mathsf{J}(\mathbf{0}, \lambda)[\boldsymbol{\varphi}^\star + \mathbf{w}] & \varepsilon = 0 \end{cases},$$

in order to determine the solutions of

$$\mathbf{F}(\mathbf{x}, \lambda) = \mathbf{0}$$

in a neighbourhood of $(\mathbf{0}, \lambda^\star)$.

5. Let $\mathbf{F} : \mathbb{R}^N \mapsto \mathbb{R}^N$ be a smooth function, with $\mathsf{J} : \mathbb{R}^N \mapsto \mathbb{R}^{N\times N}$ denoting its Jacobian matrix. Define carefully what is meant by the function $\mathbf{u}^\star : \mathbb{R} \mapsto \mathbb{R}^N$ being a periodic orbit of minimal period $T^\star > 0$ for the autonomous differential equation

$$\dot{\mathbf{u}}(t) = \mathbf{F}(\mathbf{u}(t)).$$

What is the problem of *phase indeterminacy*, and what non-trivial solution of period $T^\star$ must the linear differential equation

$$\dot{\mathbf{u}}(t) - \mathsf{J}(\mathbf{u}^\star(t))\mathbf{u}(t) = \mathbf{0}$$

possess?

Describe how, by a change of independent variable, an equation for an unknown periodic orbit and its unknown period may be constructed over the fixed interval $[0, 2\pi]$. Also explain carefully how the phase may be fixed by means of an extra scalar equation utilising a known nearby 2π-periodic function $\mathbf{v}^0 : \mathbb{R} \mapsto \mathbb{R}^N$.