1. For a Lagrangian $L : TG \mapsto \mathbb{R}$ which is left-invariant under the action of a group G, one defines Hamilton's variational principle on a family of time-dependent curves in G by

$$\delta S = \delta \int_a^b L(g(t), \dot{g}(t))\, dt = 0\,,$$

for variations $\delta g \in T_g G$ that vanish at the endpoints in time. By left-invariance of L, this Hamilton's principle is equivalent to the reduced variational principle

$$\delta S_{\rm red} = \delta \int_a^b l(\xi)\, dt = 0\,,$$

with $L(g, \dot{g}) = L(e, g^{-1}\dot{g}) = l(\xi)$ where $\xi = g^{-1}\dot{g} \in T_e G \simeq \mathfrak{g}$ is an element of the Lie algebra $\mathfrak{g}$ of the group G and $\delta\xi$ also vanishes at the endpoints in time.

(a) Compute the variation $\delta\xi = \delta(g^{-1}\dot{g})$ that is inherited from the variation $\delta g \in T_g G$.

(b) Derive the Euler-Poincaré equation from the reduced variational principle $\delta S_{\rm red} = 0$ by using the formula for $\delta\xi$ and integrating by parts.

(c) Use the Legendre transformation to obtain the corresponding Hamiltonian equation in terms of the Lie-Poisson bracket.

2. Let the vector $\mathbf{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ denote the body angular velocity of a heavy top. Let $\mathbf{\Omega}(t) = \big(R^{-1}(t)\dot{R}(t)\big)\hat{}$ denote the hat map $\hat{}: \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ where $R(t) \in SO(3)$ is a 3×3 special orthogonal matrix. The other top parameters are

- $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ is the constant moment of inertia tensor, diagonalized in the body principal axes.
- $\mathbf{\Gamma}(t) = R^{-1}(t)\hat{\mathbf{z}}$ represents the motion of the unit vector along the vertical axis, as seen from the body.
- $\boldsymbol{\chi}$ is the constant vector in the body from the point of support to the body's center of mass.
- m is the total mass of the body and g is the constant acceleration of gravity.

(a) Compute the vector equation of motion for the auxiliary variable $\mathbf{\Gamma}(t)$ from its definition.

(b) Derive the Euler-Poincaré equations for the motion of a heavy top from the following reduced action

$$\delta S_{\mathrm{red}} = 0\,, \quad \text{with} \quad S_{\mathrm{red}} = \int_a^b l(\mathbf{\Omega}, \mathbf{\Gamma})\, dt = \int_a^b \frac{1}{2}\mathbf{\Omega}\cdot\mathbb{I}\mathbf{\Omega} - mg\,\boldsymbol{\chi}\cdot\mathbf{\Gamma}\, dt,$$

where variations of $\mathbf{\Omega}$ and $\mathbf{\Gamma}$ are restricted to be of the form

$$\delta\mathbf{\Omega} = \dot{\mathbf{\Sigma}} + \mathbf{\Omega}\times\mathbf{\Sigma} \quad \text{and} \quad \delta\mathbf{\Gamma} = \mathbf{\Gamma}\times\mathbf{\Sigma}\,,$$

arising via the hat map from variations of the definitions $\mathbf{\Omega} = \big(R^{-1}\dot{R}\big)\hat{}$ and $\mathbf{\Gamma} = R^{-1}(t)\hat{\mathbf{z}}$ in which $\mathbf{\Sigma}(t) = \big(R^{-1}\delta R\big)\hat{}$ is a curve in $\mathbb{R}^3$ that vanishes at the endpoints in time.

(c) Write the formula for the Legendre transformation of a Lagrangian $l(\mathbf{\Omega}, \mathbf{\Gamma}) : \mathfrak{so}(3) \times \mathbb{R}^3 \to \mathbb{R}$ to the corresponding Hamiltonian $h(\mathbf{\Pi}, \mathbf{\Gamma}) : \mathfrak{so}(3)^* \times \mathbb{R}^3 \to \mathbb{R}$, with $\mathbf{\Pi}$ defined by $\mathbf{\Pi} = \partial l/\partial\mathbf{\Omega}$. Then differentiate the Hamiltonian and determine its partial derivatives $\partial h/\partial\mathbf{\Pi}$ and $\partial h/\partial\mathbf{\Gamma}$.

(d) Write the Euler-Poincaré equation for $\partial l/\partial\mathbf{\Omega}$ and the auxiliary equation for $\mathbf{\Gamma}$ in terms of the angular momentum variable $\mathbf{\Pi} = \partial l/\partial\mathbf{\Omega}$ and the partial derivatives of the Hamiltonian $\partial h/\partial\mathbf{\Pi}$ and $\partial h/\partial\mathbf{\Gamma}$.

(e) Recover the Lie-Poisson bracket for the heavy top expressed in terms of vectors in $\mathbf{\Pi}, \mathbf{\Gamma} \in \mathbb{R}^3$, by substituting into the time derivative of a smooth function $f(\mathbf{\Pi}, \mathbf{\Gamma})$.

3. The hat map $\widehat{\ }: \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ is given by $\widehat{\mathbf{u}} := \mathbf{u}\times$, or in Cartesian components, $(\widehat{\mathbf{v}})_{ij} = -\epsilon_{ijk}v_k$. Recall from the lectures that the hat map satisfies the following identities:

$$
\begin{aligned}
\widehat{\mathbf{u}}\mathbf{v} &= \mathbf{u}\times\mathbf{v}\,,\\
(\mathbf{u}\times\mathbf{v})\widehat{\ } &= \widehat{\mathbf{u}}\,\widehat{\mathbf{v}} - \widehat{\mathbf{v}}\,\widehat{\mathbf{u}} =: [\widehat{\mathbf{u}},\widehat{\mathbf{v}}] = [u,v]\,,\\
[\widehat{\mathbf{u}},\widehat{\mathbf{v}}]\,\mathbf{w} &= (\mathbf{u}\times\mathbf{v})\times\mathbf{w}\,,\\
\big((\mathbf{u}\times\mathbf{v})\times\mathbf{w}\big)\widehat{\ } &= \big[[\widehat{\mathbf{u}},\widehat{\mathbf{v}}]\,,\,\widehat{\mathbf{w}}\big]\,,\\
\mathbf{u}\cdot\mathbf{v} &= -\frac{1}{2}\,\mathrm{trace}(\widehat{\mathbf{u}}\,\widehat{\mathbf{v}}) =: \langle\,\widehat{\mathbf{u}}\,,\,\widehat{\mathbf{v}}\,\rangle = \langle\,u\,,\,v\,\rangle\,,
\end{aligned}
$$

in which hatting a (boldface) vector in $\mathbb{R}^3$ makes it into a 3×3 skew matrix in $\mathfrak{so}(3)$.

Recall also that the ad-action of $(\xi\,,\,\alpha)$ on $(\tilde\xi\,,\,\tilde\alpha)$ in the special Euclidean Lie algebra in three dimensions $\mathfrak{se}(3) \simeq \mathfrak{so}(3)\circledS\mathbb{R}^3$ is given by:

$$
\mathrm{ad}_{(\xi\,,\,\alpha)}(\tilde\xi\,,\,\tilde\alpha) = [(\xi,\alpha),(\tilde\xi,\tilde\alpha)] = \big([\xi\,,\,\tilde\xi\,],\,\xi\tilde\alpha - \tilde\xi\alpha\big)
$$

(a) By using the hat map rewrite this ad-action equivalently in terms of vectors in $\mathbb{R}^3$.

(b) The diamond operation $\diamond: \mathbb{R}^3\times\mathbb{R}^3 \mapsto \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is defined by

$$
\langle\beta\diamond\alpha\,,\,\tilde\xi\,\rangle \;=\; -\,\langle\beta\,,\,\tilde\xi\alpha\rangle\,.
$$

Use the hat map to rewrite the diamond operation in vector notation.
What is the interpretation of diamond as an operation among vectors in $\mathbb{R}^3$?

(c) Under the hat map, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle\,:\,se(3)^*\times se(3)\to\mathbb{R}$ transforms into the dot-product of vectors in $\mathbb{R}^3$

$$
\langle(\mu\,,\,\beta)\,,\,(\xi\,,\,\alpha)\rangle = \boldsymbol{\mu}\cdot\boldsymbol{\xi} + \boldsymbol{\beta}\cdot\boldsymbol{\alpha}
$$

Use this fact to express the ad*-action of $se(3)$

$$
\langle\mathrm{ad}^*_{(\xi\,,\,\alpha)}(\mu\,,\,\beta)\,,\,(\tilde\xi\,,\,\tilde\alpha)\rangle = \langle(\mu\,,\,\beta)\,,\,\mathrm{ad}_{(\xi\,,\,\alpha)}(\tilde\xi\,,\,\tilde\alpha)\rangle
$$

in terms of dot-products, cross-products and sums of vectors in $\mathbb{R}^3$.

(d) Consider a Lagrangian $\ell(\xi,\alpha)\;:\;\mathfrak{se}(3)\;\to\;\mathbb{R}$ whose variational derivatives are $\mu = \delta\ell/\delta\xi$ and $\beta = \delta\ell/\delta\alpha$. The corresponding $\mathfrak{so}(3)^*$ and $\mathbb{R}^3$ components of the Euler-Poincaré equation on $\mathfrak{se}(3)^*$ are given by

$$
\dot\mu = \mathrm{ad}^*_\xi\,\mu + \beta\diamond\alpha \quad \text{and} \quad \dot\beta = -\,\xi\beta\,.
$$

Use the hat map to write this equation in terms of vectors $\boldsymbol{\mu},\boldsymbol{\beta},\boldsymbol{\xi},\boldsymbol{\alpha}$ in $\mathbb{R}^3$.

What are its conservation laws in vector form?

4. (a) Write the formula for the Legendre transformation of a Lagrangian $\ell(\boldsymbol{\xi}, \boldsymbol{\alpha}) : \mathfrak{se}(3) \to \mathbb{R}$ to the corresponding Hamiltonian $h(\boldsymbol{\mu}, \boldsymbol{\beta}) : \mathfrak{se}(3)^* \to \mathbb{R}$.

   (b) Differentiate the Hamiltonian and determine its partial derivatives $\partial h / \partial \boldsymbol{\mu}$ and $\partial h / \partial \boldsymbol{\beta}$. (Assume the Lagrangian was chosen so that it allows the solution for the old variables $(\boldsymbol{\xi}, \boldsymbol{\alpha})$ in terms of the new variables $(\boldsymbol{\mu}, \boldsymbol{\beta})$.)

   (c) Determine the Lie-Poisson bracket defined on $\mathfrak{se}(3)^*$ and expressed in terms of vectors in $\mathbb{R}^3$, by rearranging the time derivative of a smooth function f.

   (d) Consider the following Lagrangian $\ell(\boldsymbol{\xi}, \boldsymbol{\alpha}) : \mathfrak{se}(3) \to \mathbb{R}$,

$$\ell(\boldsymbol{\xi}, \boldsymbol{\alpha}) = \frac{1}{2} |\boldsymbol{\xi} \times \hat{\mathbf{z}}|^2 + \frac{1}{2} \boldsymbol{\alpha} \cdot \mathbb{M} \boldsymbol{\alpha} \,,$$

   for a constant unit vector $\hat{\mathbf{z}}$ and a symmetric matrix $\mathbb{M}$.

   Write the corresponding Euler-Poincaré and Lie-Poisson Hamiltonian equations in vector form.

5. Consider Hamilton's principle $\delta S = 0$ for

$$S = \int_a^b \int_{-\infty}^{\infty} \ell(\widehat{\Omega}, \widehat{\Xi}) \, dx \, dt \,, \quad \text{with} \quad \widehat{\Omega}(t,x) = O^{-1} \partial_t O(t,x) \quad \text{and} \quad \widehat{\Xi}(t,x) = O^{-1} \partial_x O(t,x)$$

where $O(t,x) \in SO(3)$ a real valued map $O : \mathbb{R} \times \mathbb{R} \to SO(3)$.

   (a) Write the auxiliary equation for the evolution of $\widehat{\Xi}(t,x)$ obtained from its definition. Use the hat map to transform this equation into vector form for $\boldsymbol{\Xi}(t,x) \in \mathbb{R}^3$. By the same type of calculation, determine the vector forms of the variations $\delta \boldsymbol{\Omega}$ and $\delta \boldsymbol{\Xi}$.

   (b) Determine the Euler-Poincaré equation in vector form for $\boldsymbol{\Omega}(t,x) \in \mathbb{R}^3$ by using Hamilton's principle. (In the pairing use L^2 spatial integration of dot product.)

   (These auxiliary and Euler-Poincaré equations will be partial differential equations. Assume homogeneous endpoint and boundary conditions on $\boldsymbol{\Omega}(t,x)$, $\boldsymbol{\Xi}(t,x)$ and the variation $\boldsymbol{\eta} = (O^{-1} \delta O(t,x))\widehat{\ }$ when integrating by parts.)

   (c) Legendre transform this Lagrangian to obtain the corresponding Hamiltonian. Differentiate the Hamiltonian and determine its partial derivatives. Write the Euler-Poincaré equation in terms of the new momentum variable $\boldsymbol{\Pi} = \delta \ell / \delta \boldsymbol{\Omega}$.

   (d) Write equations for stationary solutions $\partial_t \to 0$ and spatially independent solutions $\partial_x \to 0$ of the Euler-Poincaré and auxiliary equation using the Legendre transformed quantities $(\boldsymbol{\Pi}, \boldsymbol{\Xi})$. How are these two types of solutions related to each other?