Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

# M4A34

# Geometry, Mechanics and Symmetry

Date: Tuesday, 30th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

 

1. The two-sphere $S^2 \subset \mathbb{R}^3$ by $S^2 = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ was shown in class to be a manifold whose two charts may be chosen as diffeomorphic stereographic coordinates projected from the North and South poles onto the equatorial plane. The coordinate charts may be taken as

(1) (valid everywhere except $z = 1$): $\xi_N = x/(1-z)\,, \qquad \eta_N = y/(1-z)\,,$

(2) (valid everywhere except $z = -1$): $\xi_S = -x/(1+z)\,, \qquad \eta_S = y/(1+z)\,.$

The sphere $S^2$ was also shown in class to be a coadjoint orbit of the Lie-Poisson bracket given in Cartesian coordinates $\mathbf{x} \in \mathbb{R}^3$ by

$$\{F, H\}(\mathbf{x}) = \frac{\partial F}{\partial x^i}\{x^i, x^j\}\frac{\partial H}{\partial x^j} = -\,\epsilon_{ijk}x^k\,\frac{\partial F}{\partial x^i}\,\frac{\partial H}{\partial x^j} = -\,\mathbf{x}\cdot\nabla F \times \nabla H\,.$$

When evaluated on the unit sphere in terms of spherical polar coordinates defined by $\cos\theta = z$ and $\tan\phi = y/x$, this Lie-Poisson bracket restricts to the canonical form

$$\{F, H\}(\theta, \phi) = -\,\frac{\partial H}{\partial \phi}\,\frac{\partial F}{\partial \cos\theta} + \frac{\partial H}{\partial \cos\theta}\,\frac{\partial F}{\partial \phi}\,.$$

(a) Write the symplectic form on $S^2$ corresponding to this canonical Poisson bracket in terms of spherical polar coordinates. What does this 2-form mean geometrically for the sphere?

(b) Let $S^1$ act on $S^2$ by rotations around the $z$-axis,

$$\Phi : S^1 \times S^2 \to S^2; \quad \Phi_\alpha(\theta, \phi) = (\theta, \phi + \alpha)\,.$$

Compute the momentum map for this action in spherical coordinates on $S^2$.

(c) Show that the manifold $S^2 \in \mathbb{R}^3$ is a Poisson manifold in stereographic coordinates, $(x, y, z) = (2\xi/(r^2+1),\ 2\eta/(r^2+1),\ (r^2-1)/(r^2+1))$ with $r^2 = \xi^2 + \eta^2$.

(d) Compute the vector field in stereographic coordinates corresponding to the momentum map for $S^1$ action on $S^2$ by rotations around the $z$-axis.

2. The formula determining the momentum map for the cotangent-lifted action of a Lie group $G$ on a smooth manifold $Q$ may be expressed in terms of the pairing $\langle \cdot \, , \, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R}$ as

$$\langle \, J \, , \, \xi \, \rangle = \langle \, p \, , \, \pounds_\xi q \, \rangle \, ,$$

where $(q,p) \in T_q^*Q$ and $\pounds_\xi q$ is the infinitesimal generator of the action of the Lie algebra element $\xi$ on the coordinate $q$.

Define appropriate pairings and determine the momentum maps explicitly for the following actions,

(a) $\pounds_\xi q = \xi \times q$ for $\mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$

(b) $\pounds_\xi q = \mathrm{ad}_\xi q$ for ad-action $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$

(c) $AqA^{-1}$ for $A \in GL(3,R)$ acting on $q \in GL(3,R)$ by matrix conjugation

(d) $Aq$ for left action of $A \in SO(3)$ on $q \in SO(3)$

(e) $AqA^T$ for $A \in GL(3,R)$ acting on $q \in Sym(3)$, that is $q = q^T$.

3. In coordinates $(a_1, a_2) \in \mathbb{C}^2$, the Hopf map $\mathbb{C}^2/S^1 \to S^3 \to S^2$ is obtained by transforming to the four quadratic $S^1$ invariant quantities

$$(a_1, a_2) \to Q_{jk} = a_j a_k^* \, , \quad \text{with} \quad j,k = 1,2 \, .$$

Let the $\mathbb{C}^2$ coordinates be expressed as $a_j = q_j + i p_j$ in terms of canonically conjugate variables satisfying the fundamental Poisson brackets $\{q_k, p_m\} = \delta_{km}$, $k, m = 1,2$.

(a) Compute the Poisson brackets $\{a_j, a_k\}$ for $j,k = 1,2$.

(b) Is the transformation $(q,p) \to (a, a^*)$ canonical? Explain why, or why not.

(c) Compute the Poisson brackets among the $Q_{jk}$, with $j,k = 1,2$.

(d) Make the linear change of variables,

$$X_0 = Q_{11} + Q_{22} \, , \quad X_1 + iX_2 = Q_{12} \, , \quad X_3 = Q_{11} - Q_{22} \, .$$

Compute the Poisson brackets among the $(X_0, X_1, X_2, X_3)$.

(e) Express the Poisson bracket $\{F(\mathbf{X}), H(\mathbf{X})\}$ in vector form among functions $F$ and $H$ of $\mathbf{X} = (X_1, X_2, X_3)$

(f) Show that the quadratic invariants $(X_0, X_1, X_2, X_3)$ themselves satisfy a quadratic relation. How is this relevant to the Hopf map?

4. (a) By using antisymmetry of contraction $X \lrcorner (X \lrcorner \alpha) = 0$ and Cartan's formula for the Lie derivative of a $k-$form $\alpha$,

$$\pounds_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha)$$

prove the following two identities:

(i) $\pounds_X d\alpha = d\big(\pounds_X \alpha\big)$

(ii) $\pounds_X (X \lrcorner \alpha) = X \lrcorner \pounds_X \alpha$

(b) Express the following in a three-dimensional Cartesian basis:

(i) $d^2 f$

(ii) $d(\mathbf{v} \cdot d\mathbf{x})$

(iii) $d(\boldsymbol{\omega} \cdot d\mathbf{S})$

(iv) $d^2(\mathbf{v} \cdot d\mathbf{x})$

5. For a Lagrangian $L : TG \mapsto \mathbb{R}$ which is left-invariant under the action of a group $G$, Hamilton's variational principle defined on a family of time-dependent curves in $G$ by

$$\delta S = \delta \int_a^b L(g(t), \dot{g}(t))\, dt = 0\,,$$

for variations $\delta g \in G$ that vanish at the at the endpoints in time. By left-invariance of $L$, this Hamilton's principle is equivalent to the reduced variational principle

$$\delta S_{\rm red} = \delta \int_a^b l(\xi)\, dt = 0\,,$$

with $L(g, \dot{g}) = L(e, g^{-1}\dot{g}) = l(\xi)$ and $\xi = g^{-1}\dot{g} \in \mathfrak{g}$ is an element of the Lie algebra $\mathfrak{g}$ of the group $G$. The variation $\delta\xi \in \mathfrak{g}$ is inherited from the variation $\delta g$ of $g \in G$; so $\delta\xi$ also vanishes at the endpoints in time.

(a) Compute the variation $\delta\xi = \delta(g^{-1}\dot{g})$ that is inherited from the variation $\delta g \in G$.

(b) Derive the Euler-Poincaré equation from the reduced variational principle $\delta S_{\rm red} = 0$.

(c) Use the Legendre transformation to obtain the corresponding Hamiltonian equation in terms of the Lie-Poisson bracket.