1. **Euler-Poincaré reduction theorem for a left-invariant Lagrangian**

(a) Write the Euler-Poincaré reduction theorem for a left $G-$invariant Lagrangian $L : TG \mapsto \mathbb{R}$ defined on a *matrix Lie group* G.

[**Hint**: The theorem says that four statements are equivalent. Define your terms when you write these four statements.]

(b) Write the Hamiltonian formulation of these Euler-Poincaré equations, as follows:

(i) Legendre transform to obtain the momentum,

(ii) obtain the Hamiltonian,

(iii) identify its variational derivative and

(iv) write the Lie-Poisson bracket.

[**Hint**: In part (ii), let the Lagrangian L be hyperregular, so the Legendre transform may be assumed to be a diffeomorphism.]

2. **Momentum maps**

(a) Consider the matrix Lie group $\mathcal{Q}$ of $n \times n$ Hermitian matrices, so that $Q^\dagger = Q$ for $Q \in \mathcal{Q}$. The Poisson (symplectic) manifold is $T^*\mathcal{Q}$, whose elements are pairs (Q, P) of Hermitian matrices. The corresponding Poisson bracket is

$$\{F, H\} = \mathrm{tr}\left(\frac{\partial F}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial H}{\partial Q}\frac{\partial F}{\partial P}\right).$$

Let G be the group $U(n)$ of $n \times n$ unitary matrices: G acts on $T^*\mathcal{Q}$ through

$$(Q, P) \mapsto (UQU^\dagger, UPU^\dagger)\,, \quad UU^\dagger = Id$$

(i) What is the linearization of this group action?

(ii) What is its momentum map?

(iii) Is this momentum map equivariant?

(b) Is the momentum map in part (a) conserved by the Hamiltonian $H = \frac{1}{2}\mathrm{tr}\, P^2$? Prove it.

3. **Ellipsoidal motions with potential energy on $GL(3,\mathbb{R})$**

Choose the Lagrangian in 3D,

$$L = \frac{1}{2}\mathrm{tr}\Big(\dot{Q}^T\dot{Q}\Big) - V\Big(\mathrm{tr}\,(Q^TQ), \det(Q)\Big),$$

where $Q(t) \in GL(3,\mathbb{R})$ is a 3×3 matrix function of time and the potential energy V is an arbitrary function of $\mathrm{tr}\,(Q^TQ)$ and $\det(Q)$.

(a) Legendre transform this Lagrangian. That is, find the momenta P_{ij} canonically conjugate to Q_{ij}. Then construct the Hamiltonian $H(Q,P)$ and write Hamilton's canonical equations of motion for this problem.

(b) Show that the Hamiltonian is invariant under $Q \to UQ$ where $U \in SO(3)$. Construct the cotangent lift of this action on P. Hence, construct the momentum map of this action.

(c) Construct another distinct action of $SO(3)$ on this system which also leaves its Hamiltonian $H(Q,P)$ invariant. Construct its momentum map. Do the two momentum maps Poisson commute? Why?

4. **$GL(n,\mathbb{R})-$invariant motions**

Consider the Lagrangian

$$L = \frac{1}{2}\,\mathrm{tr}\Big(\dot{S}S^{-1}\dot{S}S^{-1}\Big) + \frac{1}{2}\,\dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}}\,,$$

where S is an $n\times n$ symmetric matrix and $\mathbf{q} \in \mathbb{R}^n$ is an $n-$component column vector.

(a) Legendre transform to construct the corresponding Hamiltonian and canonical equations.

(b) Show that both Lagrangian and the Hamiltonian the system are invariant under the group action

$$\mathbf{q} \to G\mathbf{q} \quad \text{and} \quad S \to GSG^T$$

for any constant invertible $n\times n$ matrix, G.

(c) Compute the infinitesimal generator for this group action and construct its corresponding momentum map. Is this momentum map equivariant?

(d) Verify directly that this momentum map is a conserved $n\times n$ matrix quantity by using the equations of motion.

5. **EPDiff equation for the H^1 metric** The EPDiff(H^1) equation is obtained from the Euler-Poincaré reduction theorem for a right-invariant Lagrangian, when one defines this Lagrangian to be half the H^1 norm on the real line of the vector field of velocity u, namely,

$$l(u) = \frac{1}{2}\|u\|^2_{H^1} = \frac{1}{2}\int_{-\infty}^{\infty} u^2 + u_x^2 \, dx \,.$$

(Assume u satisfies homogeneous boundary conditions.)

(a) Write the EPDiff(H^1) equation on the real line in terms of its velocity u and its momentum $m = \delta l/\delta u$ in one spatial dimension.

(b) Write the traveling wave solution of EPDiff(H^1) for its velocity, $u(x - ct)$. (Hint: This is the one-peakon solution.) Write the momentum $m(x - ct)$ for the one-peakon solution.

(c) Write the $N-$peakon singular momentum solution $m(x, t)$ of EPDiff(H^1) in terms of canonically conjugate variables q_i and p_i, with $i = 1, 2, \ldots, N$.

(d) For the 2-peakon dynamics of EPDiff(H^1),

(i) Write the canonical equations for the conserved Hamiltonian $H(q, p, Q, P)$ in terms of sum and difference variables

$$P = p_1 + p_2\,, \quad Q = q_1 + q_2\,, \quad p = p_1 - p_2\,, \quad q = q_1 - q_2$$

(ii) Consider the scattering of two peakons that are initially well separated and have speeds c_1 and c_2 with $c_1 > c_2$, so that they collide.

Solve the canonical equations of 2-peakon dynamics for $q(t)$ and $p(t)$, in the perfectly anti-symmetric case $Q = 0$, for the "head-on" case $P = 0$ with $c_1 = -c_2 = c$, so that the collision occurs at time $t = 0$ at the origin $x = 0$. How does the momentum difference p behave at the moment of collision?