Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

# MSA4/M4A34 Geometry, Mechanics and Symmetry

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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**1.** Define the term *manifold*. Explain what is meant by the *tangent bundle* of a manifold. Derive the formulae transforming tangent vectors between 2 different sets of coordinates $\{q^i\}_{i=1}^N$ and $\{Q^i\}_{i=1}^N$ on an $N$-dimensional manifold, if

$$q^i = f^i(\mathbf{Q})$$

Derive also the formula for transforming covectors between the same two coordinate systems.

A particle moves on the 2-sphere $|\mathbf{x}| = 1$. Its Lagrangian is

$$L = \frac{1}{2}|\dot{\mathbf{x}}|^2 - V(\mathbf{x}).$$

Write this Lagrangian in spherical polar coordinates and derive the Euler-Lagrange equations. Why is it not correct to derive the Euler-Lagrange equations corresponding to the coordinates $(x^1, x^2, x^3)$ from $L$ directly?

**2.** List the defining properties for a Poisson bracket. A Poisson manifold has Poisson bracket relations between its coordinates $x_i$ given by:

$$\{x_i, x_j\} = B_{ij}(\mathbf{x}).$$

Write down all the conditions which the $B_{ij}(\mathbf{x})$ must satisfy.

If the $B_{ij}(\mathbf{x})$ are linear, given by

$$B_{ij}(\mathbf{x}) = C_{ij}^k x_k$$

find the conditions satisfied by the constants $C_{ij}^k$. State where such constants can be found.

A Poisson bracket on $\mathbb{R}^3$ is given by

$$\{x_1, x_2\} = 0,$$

$$\{x_1, x_3\} = x_2,$$

$$\{x_2, x_3\} = -x_1.$$

Write down Hamiltons's equations corresponding to a Hamiltonian $H(x_1, x_2, x_3)$, and find a Casimir for this bracket.

**3.** Explain what is meant by an *a*ction of a Lie group on a manifold $M$. Explain also the idea of an action of a Lie algebra $\mathfrak{g}$ on $M$ and the relationship between these two concepts.

A system has Lagrangian

$$L = \frac{1}{2}\left((\dot{U}U^{-1}(\dot{U}U^{-1}\right) - \frac{1}{2}\left(KUU^{T}\right).$$

Here $U$ is a $3 \times 3$ invertible matrix and $K$ is a constant $3 \times 3$ matrix. Find the Hamiltonian for the system.

Show that the Lagrangian is invariant under transformations of the form

$$\phi_V : U \to UV^{-1}$$

if $V$ is any orthogonal matrix, and show that this is an action of $O(3)$ on the system. Find the corresponding Lie algebra action, and calculate its momentum map. State, without directly calculating them, the Poisson bracket relations between different components of the momentum map.

**4.** Explain the steps in reducing the phase space of a Hamiltonian system which is invariant under the action of a Lie group $G$. Illustrate this by carrying out the procedure for the Hamiltonian

$$H = \frac{1}{2}|\mathbf{p}|^2 + \frac{1}{2}|\mathbf{x}|^2,$$

where $\mathbf{p}$ and $\mathbf{x}$ take values in $\mathbb{R}^2$. Hence obtain an explicit solution, in elementary functions, to the motion of the simpler Hamiltonian

$$\widetilde{H} = \frac{1}{2}P^2 + \frac{1}{2}R^2 + \frac{1}{2}\frac{\mu^2}{R^2}.$$

**5.** Explain what is meant by a completely integrable Hamiltonian system.
If $L, M$ are both $n \times n$ matrices, and $L$ satisfies the 'Lax equation':

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [M, L],$$

show that the quantities

$$H_k = \frac{1}{k}\left(L^k\right)$$

are all constant in time.
The Toda equations are written in Lax form with

$$L = \begin{pmatrix} p_1 & \exp(x_1 - x_2) & 0 \\ 1 & p_2 & \exp(x_2 - x_3) \\ 0 & 1 & p_3 \end{pmatrix}.$$

and

$$M = \begin{pmatrix} p_1 & 0 & 0 \\ 1 & p_2 & 0 \\ 0 & 1 & p_3 \end{pmatrix}.$$

Calculate the equations of motion, and show that they are consistent with Hamilton's equations generated by $H_2$ and the canonical Poisson bracket. Hence find 3 independent conserved quantities, and show that the Toda equations are completely integrable.