1. Blood flow in the body is driven by a time-periodic pressure pulse from the heart whose steady part is much smaller than the alternating part. It is physiologically desirable that the flux down the arteries should always be positive. Consider the simple 2-D model below:

Flow is driven in the rigid channel $a > y > -a$ by the oscillating pressure gradient

$$-\frac{\partial p}{\partial x} = G_0 + G_1 \cos \Omega t \ ,$$

where G_0, G_1 and Ω are constants with $G_1 \gg G_0 > 0$.
Seek a unidirectional solution to the incompressible Navier-Stokes equations of the form $\mathbf{u} = (u(y,t),\, 0,\, 0)$ with

$$u = u_0(y) + \Re e \left[u_1(y) e^{i\Omega t} \right] \ ,$$

where $\Re e$ denotes the real part.

Find u_0 and show that for a suitable real constant δ,

$$u_1 = \frac{G_1}{\rho i \Omega} \left[1 - \frac{\cosh[(1+i)y/\delta]}{\cosh[(1+i)a/\delta]} \right] \ .$$

As $\Omega \to \infty$, show that the wall shear stress

$$\mu \left. \frac{\partial u}{\partial y} \right|_{y=a} \quad \to \quad -aG_0 - G_1 \left(\frac{\mu}{2\rho\Omega} \right)^{1/2} \left[\cos \Omega t + \sin \Omega t \right] \ .$$

Discuss whether or not the velocity can be negative for some values of y and t, and comment on the implications for blood flow.

2. A two-dimensional, alternating magnetic field $\mathbf{B}(x,\, y,\, t)$ is represented by

$$\mathbf{B} = \Re e\left[\mathbf{B}'(x,y)e^{i\omega t}\right] \qquad \text{where } \Re e \text{ denotes the real part.}$$

Show that the governing equations in a region of constant conductivity σ and permeability μ_0,

$$\nabla\wedge\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\wedge\mathbf{B} = \mu_0\mathbf{j}, \qquad \nabla\cdot\mathbf{B} = 0, \qquad \mathbf{j} = \sigma\mathbf{E}\,,$$

are satisfied if

$$\nabla^2\mathbf{B}' = i\omega\mu_0\sigma\mathbf{B}'\ .$$

Insulating gas occupies $y > 0$ while metal with conductivity σ occupies $y < 0$. As $y \to +\infty$, $\mathbf{B} \to (B_0\cos\omega t,\, 0,\, 0)$, for constant B_0 and $\mathbf{B} \to (0,\, 0,\, 0)$ as $y \to -\infty$.
Find $\mathbf{B}'$ inside and outside the metal.

The time-averaged heating rate per length in the x-direction is defined by

$$\overline{W} = \int_{-\infty}^{0} \frac{|j'|^2}{2\sigma}\, dy\ ,$$

where $\mathbf{j} = \Re e\left[(0,\, 0,\, j'e^{i\omega t})\right]$. Calculate $\overline{W}$ and show also that $\nabla\wedge(\mathbf{j}\wedge\mathbf{B}) = 0$.

From these results, summarise the effect of an oscillating uniform horizontal field on an initially stationary bath of liquid metal:

(i) How does the fluid move, if at all?

(ii) Is there any qualitative difference between high frequency ($\omega \to \infty$) and high conductivity ($\sigma \to \infty$)?

3. Give an account of any topic covered in the course not explicitly examined on this paper. (For example: solidification or melting, surfactant transport, flow in curved pipes, shear-enhanced dispersion, electrically charged drops etc.)

   Discuss, as appropriate, the practical significance of the problem, the assumptions of the model and the solution.

4. (a) Consider the heat equation for a temperature field, $T(x,t)$,

$$T_t = \alpha T_{xx}$$

for $0 \le x < \infty$ and $t \ge 0$, where α is a constant. Show that this equation allows for similarity solutions of the form

$$T(x,t) = A \operatorname{erfc}\left(\frac{x}{2(\alpha t)^{\frac{1}{2}}}\right) + B$$

for constants A and B. For what boundary and initial conditions is such a solution suitable?

(b) Using the above representation for the temperature field, or otherwise, consider the following temperature problem involving a phase change:

The temperature field, $T(x,t)$, satisfies

$$T_t = \alpha T_{xx}$$

for $s(t) \le x < \infty$ and $t \ge 0$. In the region $x < s(t)$ the temperature is fixed at $T = T_m$ always. As $x \to \infty$ and $t = 0$ we have $T = T_0$, where T_0 is constant, and at $x = s(t)$ we set $T = T_m$ (where T_m is constant and $T_0 < T_m$). On $x = s(t)$ the Stefan boundary condition

$$-kT_x = \rho L \frac{ds}{dt}$$

is taken. At $t = 0$ the phase change boundary $s(t)$ is at $x = 0$ (i.e. $s(0) = 0$). In this problem is the material in $x > s(t)$ about to melt or solidify?

Find the temperature field $T(x,t)$ and the position of the phase change boundary $s(t)$ in terms of a constant λ that satisfies a transcendental equation that you should derive.

$$\left[\text{You are given that} \quad \operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} \exp(-\tau^2) \mathrm{d}\tau\right]$$

5. If we consider a surfactant-driven thin fluid layer flow in axisymmetry, then the fluid layer height, $h(r,t)$, and surfactant concentration, $\Gamma(r,t)$, satisfy to leading order the evolution equations

$$\frac{\partial h}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(rh^2\frac{\partial \Gamma}{\partial r}\right) = 0$$

$$\frac{\partial \Gamma}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma h\frac{\partial \Gamma}{\partial r}\right) = 0.$$

Let us adopt the constraint that the mass M initially deposited at the origin satisfies

$$M = 2\pi \int_0^\infty r\Gamma(r,t)dr$$

for all time, where M is constant.

(a) Adopting the similarity variables

$$\xi = \frac{r}{\xi_s t^a}, \quad \Gamma(r,t) = \frac{\xi_s^2 G(\xi)}{t^b}, \quad h(r,t) = H(\xi)$$

where a and b are constants to be determined and ξ_s is a constant chosen such that, in these rescaled variables, the surfactant leading edge (the position separating the surfactant laden interface from surfactant-free interface) is at $\xi = 1$.

(b) Find the powers a and b and therefore deduce the power of time at which the surfactant leading edge progresses. Deduce ODEs that H and G satisfy and show that

$$H(\xi) = 2\xi^2, \quad G(\xi) = -\frac{1}{8}\log \xi$$

satisfy the ODEs. Hence, or otherwise, deduce ξ_s.