Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

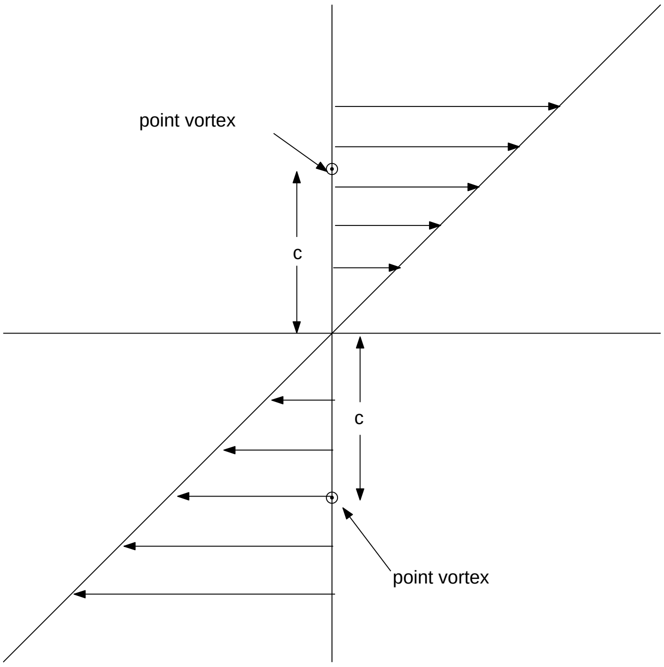
M4A32

Vortex dynamics

Date: Monday, 22nd May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.



1. (a) Consider a two-dimensional shear flow of an ideal fluid, in an (x, y)-plane, in which the velocity (u, v) is given by

$$(u, v) = (Uy, 0)$$

where $U > 0$ is some constant. Suppose that two point vortices, each of circulation $\Gamma > 0$, are now placed in this flow at points $(0, \pm c)$ where $c > 0$ is some constant. A sketch of the configuration is in the figure.

(i) Find an expression for the streamfunction for this flow as a function of z and $\bar{z}$ where $z = x + iy$.

(ii) Find the relationship between the parameters U, Γ and c for the configuration to be in steady equilibrium.

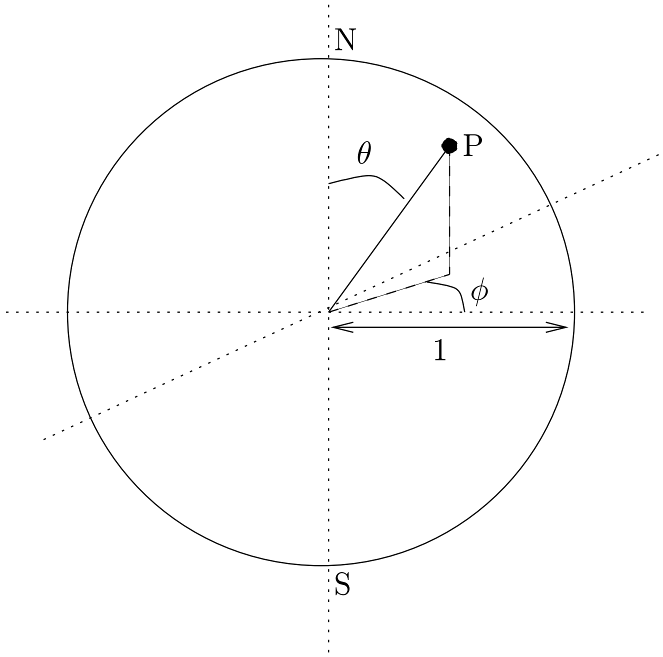
(b) Consider an elliptical vortex patch, with constant vorticity ω_0, centred at the origin $z = 0$ in a flow which has the form

$$u - iv = \epsilon z + \delta \bar{z}$$

as $|z| \to \infty$. Here, ϵ and δ are two complex numbers. A conformal map from the interior of a unit ζ-disc to the fluid region exterior to the ellipse is given by

$$z(\zeta) = \frac{\alpha}{\zeta} + \beta \zeta$$

where α is a real constant and β is a complex constant. Find an expression for the instantaneous velocity field *inside* the patch in terms of $\omega_0, \epsilon, \delta, \alpha$ and β.



2. Let a typical point P on a unit-radius sphere have spherical polar coordinates $(1, \theta, \phi)$ where θ is the (latitudinal) angle measured from the north pole N and ϕ is the azimuthal (longitudinal) angle (see figure above).

   (a) Describe the construction of the stereographic projection of points on the sphere onto a complex ζ-plane through the equator with $\zeta = 0$ corresponding to the south pole S of the sphere. Write down an expression for the projected point ζ in terms of θ and ϕ.

   (b) Write down the expression for the streamfunction associated with a single unit-circulation point vortex situated at the south pole S. Is this configuration an equilibrium?

   (c) Now consider the construction of a more complicated equilibrium. In addition to the unit-circulation point vortex at S, the entire *southern* hemisphere is covered with uniform vorticity but the northern hemisphere is taken to be a region of stagnant flow. Bearing in mind the constraint that the total vorticity on the surface of a sphere must be zero, write down an expression for the streamfunction (in terms of the stereographically-projected coordinate ζ) that is valid in the southern hemisphere.

   (d) Verify that your solution is an equilibrium by checking that the correct boundary conditions hold at the vortex jump at the equator.

3. Consider the motion of a single point vortex in the unit disc $|\zeta| \leq 1$. The circular boundary $|\zeta| = 1$ is an impenetrable barrier for the flow. The flow is incompressible. Apart from the single point vortex, the flow is irrotational.

(a) Let the position of the point vortex be at $\zeta = \alpha$. Define the Green's function $G(\zeta; \alpha, \bar{\alpha})$ associated with point vortex motion in this domain (that is, explain the boundary value problem satisfied by this Green's function).

(b) Verify that the function

$$G(\zeta; \alpha, \bar{\alpha}) = -\frac{1}{2\pi} \log \left| \frac{\zeta - \alpha}{\alpha(\zeta - \bar{\alpha}^{-1})} \right|$$

satisfies all the conditions required of the Green's function $G(\zeta; \alpha, \bar{\alpha})$ defined in part (a).

(c) The formula for the Hamiltonian $H^{(\zeta)}(\alpha, \bar{\alpha})$ governing the motion of a single point vortex of unit circulation in the unit disc is

$$H^{(\zeta)}(\alpha, \bar{\alpha}) = \frac{1}{2} g(\alpha; \alpha, \bar{\alpha})$$

where $g(\zeta; \alpha, \bar{\alpha})$ is defined by the equation

$$G(\zeta; \alpha, \bar{\alpha}) = -\frac{1}{2\pi} \log |\zeta - \alpha| + g(\zeta; \alpha, \bar{\alpha}).$$

Find an explicit expression for $H^{(\zeta)}(\alpha, \bar{\alpha})$.

(d) Find a conformal mapping $z(\zeta)$ from the interior of the unit ζ-disc to the unbounded region in a complex z-plane (where $z = x + iy$) exterior to an infinite straight wall along the x-axis having a gap, of unit length, between $-1/2 \leq x \leq 1/2$.

(e) By using the result that the Hamiltonian $H^{(z)}(z_\alpha, \bar{z}_\alpha)$ for the motion of a point vortex at position z_α in the domain introduced in part (d) is given by

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = H^{(\zeta)}(\alpha, \bar{\alpha}) + \frac{1}{4\pi} \log \left| \frac{dz}{d\zeta}(\alpha) \right|$$

where $z_\alpha = z(\alpha)$, show that the equation for the *critical* trajectory – that is, the one separating trajectories where the vortex goes through the gap from those trajectories that by-pass the gap – is given by

$$|1 - \alpha(z)\overline{\alpha(z)}||1 - \alpha(z)^2| = |1 + \alpha(z)^2|^2.$$

where

$$\alpha(z) = \frac{1}{2z} \left(1 - \sqrt{1 - 4z^2} \right).$$

4. (a) Show that in a two-dimensional incompressible flow of an ideal fluid, the vorticity ω is related to the streamfunction ψ by means of the relation

$$\omega = -\nabla^2 \psi.$$

(b) Show also that any two-dimensional flow in which $\omega = h(\psi)$, where h is an arbitrary differentiable function of ψ, is a steady solution of the vorticity equation.

(c) Now let $h(\psi) = -4e^{-2\psi}$. Verify that a formal solution of the steady Euler equations is given by

$$\psi(z, \bar{z}) = -\frac{1}{2} \log \left(\frac{f'(z)\bar{f}'(\bar{z})}{(1 + f(z)\bar{f}(\bar{z}))^2} \right)$$

where $f(z)$ is an arbitrary analytic function of z.

(d) Using the formula in part (c), show that if $f(z)$ has a simple pole singularity at some point a so that, near $z = a$,

$$f(z) = \frac{b}{z - a} + \text{analytic},$$

where a and b are complex numbers, then the associated streamfunction is *non*-singular at $z = a$.

(e) Now pick

$$f(z) = \frac{a}{z^N} + b$$

where $N \geq 2$ is a positive integer so that $f(z)$ has an N-th order pole at $z = 0$. Show that the associated streamfunction, as given in part (c), has a point vortex singularity at $z = 0$ and find the circulation of this point vortex.

5. Let D be some two-dimensional, simply connected region of non-zero vorticity. It is proposed that a possible equilibrium solution of the Euler equations is given by a streamfunction of the form

$$\psi(z, \bar{z}) = \begin{cases} -\frac{\omega_0}{4}\left[z\bar{z} - F(z) - \overline{F(z)}\right] & z \in D \\ 0 & z \notin D \end{cases}$$

where $z = x + iy$, ω_0 is a constant and $F(z)$ is given by the indefinite integral

$$F(z) = \int^z S(z')dz'$$

where $S(z)$ is defined to be the analytic function that equals $\bar{z}$ everywhere on ∂D, that is,

$$S(z) = \bar{z}, \quad \text{on } \partial D.$$

∂D denotes the boundary of the region D.

(a) If (u, v) are the Cartesian components of the velocity field, show that, inside D,

$$u - iv = -\frac{i\omega_0}{2}\left(\bar{z} - S(z)\right).$$

Hence show that the vorticity distribution inside D is uniform except possibly at any singularities of the analytic function $S(z)$.

(b) Show that $\psi(z, \bar{z})$ satisfies all the required boundary conditions at the vortex jump ∂D.

(c) Now suppose that a one-to-one conformal mapping from the unit ζ-disc $\{|\zeta| \leq 1\}$ to the region D is given by

$$z(\zeta) = \zeta + \frac{b\zeta}{\zeta^2 - a^2}$$

where a and b are some real constants and $|a| > 1$. Show that, in terms of the variable ζ, the function $S(z)$ is given by the formula

$$S(z) = \frac{1}{\zeta} + \frac{b\zeta}{1 - a^2\zeta^2}.$$

(d) Hence show that $S(z)$ in fact has three simple pole singularities inside D and find explicit formulae for the positions of these point vortices in terms of a and b.

(e) What kind of flow singularities do these three simple poles of $S(z)$ correspond to? Hence describe what additional conditions on the parameters a and b must be satisfied at each of these three singularities if $\psi(z, \bar{z})$ is to represent a consistent equilibrium of the Euler equations. (Note: it is not necessary to find explicit formulae for these additional conditions – just describe how they might be derived in principle).