Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

and MSc EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M4A32/MSA2 Vortex Dynamics

Date: Friday 20th May 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider the incompressible fluid motion associated with two point vortices in the plane. One point vortex with circulation Γ is at the complex position $z_1(t)$, a second point vortex with circulation 2Γ is at $z_2(t)$. Apart from the vortices, the flow is otherwise irrotational. Initially, at $t = 0$, the two vortices are separated by unit distance.

   (a) Find the complex potential $w(z,t)$ associated with the flow.

   (b) Write down the ordinary differential equations satisfied by $z_1(t)$ and $z_2(t)$.

   (c) Show that the separation of the point vortices is a constant of the motion.

   (d) Show that $z_0 = \frac{1}{2}z_1(t) + z_2(t)$ is also a constant of the motion.

   (e) By considering the change of variables

$$\mathcal{Z}_1(t) = z_1(t) - z_0,$$
$$\mathcal{Z}_2(t) = z_2(t) - z_0,$$

   and by seeking solutions of the form

$$\mathcal{Z}_1(t) = r_1 e^{i\theta(t)}, \quad \mathcal{Z}_2(t) = -r_2 e^{i\theta(t)}$$

   where r_1 and r_2 are positive constants, show that the vortices rotate with constant angular velocity about the point z_0. Find the angular velocity.

2. Two point vortices, each of strength κ, are equally-spaced around the latitude circle $\theta = \theta_0$ on the surface of a unit-radius sphere, where θ is the usual polar angle in spherical polar coordinates. $\theta = 0$ corresponds to the north pole of the sphere, $\theta = \pi$ corresponds to the south pole and it is assumed that $0 < \theta_0 < \pi$. Apart from the point vortices, the incompressible fluid motion on the spherical surface is irrotational.

   (a) The two point vortices rotate with constant angular velocity Ω about the axis through the north and south poles of the sphere. Find Ω as a function of κ and θ_0.

   (b) An additional point vortex of strength κ_s is now placed at the south pole. As a result, the angular velocity of the two point vortices doubles so that it is now equal to 2Ω. Find κ_s as a function of κ and θ_0.

   (c) A fourth point vortex of strength κ_n is now added at the north pole. As a result, the combined configuration of four point vortices is now completely stationary (i.e., none of the vortices move). Find κ_n as a function of κ and θ_0.

[*Hint:* you may use the fact that if the flow has streamfunction ψ and the velocity field in spherical polar coordinates is $(0, u_\theta, u_\phi)$ then

$$u_\phi - iu_\theta = \frac{2\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}\bigg|_{\bar{\zeta}}$$

where $\psi = \psi(\zeta, \bar{\zeta})$ and ζ is the usual stereographically-projected complex coordinate onto a plane through the equator, i.e.,

$$\zeta = \cot(\theta/2)e^{i\phi}$$].

3. This question concerns the motion of a single point vortex in the unit disc $|\zeta| \leq 1$. The circular boundary $|\zeta| = 1$ is an impenetrable barrier for the flow. The flow is incompressible. Apart from the single point vortex, the flow is irrotational.

(a) Let the position of the point vortex be at $\zeta = \alpha$. Define the Green's function $G(\zeta; \alpha, \bar{\alpha})$ associated with point vortex motion in this domain (that is, explain the boundary value problem satisfied by this Green's function).

(b) Verify that the function

$$G(\zeta; \alpha, \bar{\alpha}) = -\frac{1}{2\pi} \log \left| \frac{\zeta - \alpha}{\alpha(\zeta - \bar{\alpha}^{-1})} \right|$$

satisfies all the conditions required of the Green's function $G(\zeta; \alpha, \bar{\alpha})$ defined in part (a).

(c) The formula for the Hamiltonian $H(\alpha, \bar{\alpha})$ governing the motion of a point vortex of unit circulation in the unit disc is

$$H(\alpha, \bar{\alpha}) = \frac{1}{2} g(\alpha; \alpha, \bar{\alpha})$$

where $g(\zeta; \alpha, \bar{\alpha})$ is defined by the equation

$$G(\zeta; \alpha, \bar{\alpha}) = -\frac{1}{2\pi} \log |\zeta - \alpha| + g(\zeta; \alpha, \bar{\alpha}).$$

Find an explicit expression for $H(\alpha, \bar{\alpha})$.

(d) What are the trajectories of the point vortex?

(e) Let $\alpha = x + iy$. Show that $H(\alpha, \bar{\alpha})$ satisfies the partial differential equation

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right) H = -\frac{1}{\pi} e^{-8\pi H}.$$

4. Consider the motion of a point vortex in the upper-half ζ-plane where the real ζ-axis acts as an impenetrable barrier for the flow. The Hamiltonian $H(\alpha, \bar{\alpha})$ governing the motion of a single point vortex of circulation Γ where α is the (complex) point vortex position is

$$H(\alpha, \bar{\alpha}) = \frac{\Gamma^2}{4\pi} \log |\alpha - \bar{\alpha}|.$$

(a) What are the point vortex trajectories?

(b) It is known that if the Hamiltonian in a domain D_ζ in a ζ-plane is $H^{(\zeta)}(\alpha, \bar{\alpha})$ then the Hamiltonian $H^{(z)}(z_\alpha, \bar{z}_\alpha)$ governing the motion in the region D_z obtained by the one-to-one conformal mapping of D_ζ by the function $z(\zeta)$ is

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = H^{(\zeta)}(\alpha, \bar{\alpha}) + \frac{\Gamma^2}{4\pi} \log |z_\zeta(\alpha)|$$

where $z_\alpha = z(\alpha)$. By using this result, find an explicit expression for the Hamiltonian $H^{(z)}(z_\alpha, \bar{z}_\alpha)$ for the motion of a point vortex of circulation Γ at a point z_α in the first quadrant of a complex z-plane, i.e., in the domain $\mathrm{Re}[z] > 0, \mathrm{Im}[z] > 0$.

(c) Show that the vortex trajectories are the curves

$$r \sin 2\theta = \text{constant}$$

where (r, θ) are the usual polar coordinates.

(d) Suppose that, at some instant, the point vortex is at the point $z = 1 + i$. Find the instantaneous velocity of the fluid at the point $z = 1$.

5. Consider an isolated, finite-area vortex patch with uniform vorticity ω_0. At some instant, the fluid exterior to the patch is in irrotational motion decaying at infinity while the shape of the fluid region exterior to the patch is given by the image of the unit disc in a parametric ζ-plane, i.e., $|\zeta| \leq 1$, under the conformal mapping

$$z(\zeta) = \frac{1}{\zeta} + b\zeta^2$$

where b is some real constant.

(a) If (u, v) are the Cartesian components of the fluid velocity, show that the complex velocity inside the patch is given by

$$u - iv = -\frac{i\omega_0}{2}\left(\bar{z} - bz^2\right).$$

(b) Show that, as $z \to \infty$, the complex velocity decreases to zero according to

$$u - iv \sim \frac{i\omega_0}{2z}(2b^2 - 1).$$

(c) By using the result from part (b) show that, at large distances, the velocity field generated by the vortex patch resembles that generated by a point vortex at $z = 0$ having circulation $\pi\omega_0(1 - 2b^2)$.