Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MSc EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M4A31/MSA1

High Reynolds Number Flows and Boundary Layer Theory

Date: Wednesday 24th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. For the singular perturbation problem

$$\epsilon y'' + (x+1)y' + y^2 = 0 \quad (0 < \epsilon \ll 1), \quad y(0) = 0, \quad y(1) = 1,$$

seek an outer expansion

$$y(x) = y_0(x) + \epsilon y_1(x) + \cdots \ .$$

(i) Give a brief argument to determine the location of the inner layer and the boundary condition to be satisfied by y_0, and hence determine the leading-order solution $y_0(x)$.

Seek an inner expansion

$$y(x) = Y_0(X) + \epsilon Y_1(X) + \cdots \ ,$$

where X is an appropriate inner variable that you need to identify, and find the leading-order solution Y_0.

(ii) Use the leading-order outer and inner solutions to construct a composite solution $C_{00}y$ based on the additive rule.

Sketch the outer, inner and composite solutions, and indicate the respective regions in which these solutions are validity.

(iii) Suppose that the interval is changed to $-1 \le x \le 1$ so that the boundary condition becomes $y(-1) = 0$. Determine the appropriate inner variable and the order-of-magnitude of the inner solution. (You are not required to solve the inner problem.)

2. Consider the unsteady boundary-layer equations

$$U_x + V_Y = 0, \quad U_t + UU_x + VU_Y = u_{0,t} + u_0 u_{0,x} + U_{YY}, \tag{1}$$

for $U(x, Y)$ and $V(x, Y)$, where $u_0(x, t)$ is the inviscid slip velocity.

(i) Show that if

$$\begin{aligned} \widetilde{U}(x, Y, t) &= U\Big(x,\, Y + f(x, t)\Big), \\ \widetilde{V}(x, Y, t) &= V\Big(x,\, Y + f(x, t)\Big) - f_x(x, t) U\Big(x,\, Y + f(x, t)\Big) - f_t(x, t), \end{aligned}$$

$\widetilde{U}$ and $\widetilde{V}$ also satisfy the same boundary-layer equations.

Explain how the above relations could be used to solve the boundary-layer flow along a curved surface $Y = -f(x, t)$, on which the boundary conditions, $U = 0$ and $V = -f_t(x, t)$, have to be satisfied.

(ii) Let $\chi = u_0^2 - U^2$, and ψ be the stream function such that

$$U = \psi_Y, \quad V = -\psi_x\,.$$

Show that the *steady* version of boundary-layer equations (**??**) can be written as

$$\frac{\partial \chi}{\partial x} = U \frac{\partial^2 \chi}{\partial \psi^2}\,, \qquad Y = \int_0^{\psi} \frac{\mathrm{d}\psi}{(u_0^2 - \chi)^{1/2}}\,.$$

Comment on the nature of the boundary-layer equations, and the implication if $U < 0$ for some ψ.

3. A wall jet over a flat plate is subjected to a weak steady suction through a slot at a distance L from the leading edge. In a coordinate system (x, y), where x and y are normalised by L, the flow field is governed by the two-dimensional steady Navier-Stokes equations

$$\left.\begin{aligned} u_x + v_y &= 0, \\ uu_x + vu_y &= -p_x + \tfrac{1}{R}(u_{xx} + u_{yy}), \\ uv_x + vv_y &= -p_y + \tfrac{1}{R}(v_{xx} + v_{yy}). \end{aligned}\right\}$$

The slot has a width of $O(D) \ll 1$, and the suction velocity is given by

$$v = \epsilon_s V_s(X), \quad \text{with} \quad X = x/D,$$

where $\epsilon_s \ll 1$. The suction only produces a small perturbation to the oncoming wall-jet flow so that in the main part of the boundary layer the flow field can be written as

$$(u, v, p) = (U_0(x, Y), R^{-\frac{1}{2}}V_0, P_0) + \epsilon\Big(U, \frac{R^{-\frac{1}{2}}}{D}V, \frac{R^{-1}}{D^2}P\Big),$$

where $\epsilon \ll O(1)$ is to be determined later. The wall jet profile, U_0, has the property that

$$U_0(x, Y) \to Y \quad \text{as} \quad Y \to 0, \qquad \text{and} \quad U_0(x, Y) \to 0 \quad \text{as} \quad Y \to \infty,$$

with $Y = R^{\frac{1}{2}}y$.

(i) Derive the governing equations for (U, V, P), and verify that they have the solution

$$U = A(X)\, U_{0,Y}\,, \quad V = -A'(X)\, U_0, \quad P = A''(X) \int_0^Y U_0^2 \, \mathrm{dY} + \mathrm{P_0(X)},$$

where $A(X)$ and P_0 are arbitrary functions of X.
Explain briefly why P_0 must be chosen so that $P \to 0$ as $Y \to \infty$.

(ii) Explain why it is necessary to introduce a viscous sub-layer (lower deck). Deduce the width of this layer, Δ, in terms of D and R.
Show that

$$\epsilon \sim \epsilon_s R^{\frac{1}{2}} D^{\frac{2}{3}}.$$

Estimate the inertia in the viscous sub-layer.
Deduce that the flow becomes interactive if

$$D = O(R^{-\frac{3}{7}}).$$

(iii) Show that if $\epsilon_s \sim R^{-\frac{5}{14}}$, the lower deck is nonlinear.
Write down the equations governing the lower deck.
State the pressure-displacement relation as well as the boundary and matching conditions.

4. The interaction of a (Mach) wave with a supersonic boundary-layer flow may be described by triple deck theory, in which the lower deck is governed by the equations

$$u_X + v_y = 0, \quad uu_X + vu_y = -p_X + u_{yy},$$

with the boundary conditions $u = v = 0$ on $y = 0$,

and the matching condition with the main deck

$$u \to y + A(X) \quad \text{as} \quad y \to \infty \ .$$

The pressure p is related to the upper deck pressure $\overline{p}$ via

$$p = \overline{p}(X, 0) + P_I(X),$$

where $P_I(X)$ is a given function representing the pressure of the incident wave. In the upper deck, the pressure of the reflected wave, $\overline{p}(X, \overline{y})$, is governed by equation

$$(M^2 - 1)\frac{\partial^2 \overline{p}}{\partial X^2} - \frac{\partial^2 \overline{p}}{\partial \overline{y}^2} = 0, \qquad (*)$$

subject to the boundary conditions

$$\frac{\partial \overline{p}}{\partial \overline{y}} = A''(X) - (M^2 - 1)^{1/2} P_I'(X), \quad \text{on} \quad \overline{y} = 0 \ , \qquad \overline{p} \text{ is finite as } \overline{y} \to \infty \ ,$$

where $M > 1$ is the Mach number.

(i) Verify that

$$\overline{p} = \overline{f}\Big(X - (M^2 - 1)^{1/2}\overline{y}\Big)$$

satisfies equation $(*)$ for an arbitrary function $\overline{f}$, and derive the pressure-displacement relation which relates p, A and P_I.

(ii) For the case where $P_I(X) = \epsilon\, \mathrm{e}^{\mathrm{i}\alpha \mathrm{X}} + \text{c.c.}$ with $\epsilon \ll O(1)$, seek a solution of the form

$$(u, v, p, A) = (y, 0, 0, 0) + \epsilon\Big(\widehat{u}(y), \widehat{v}(y), \widehat{p}, \widehat{A}\Big)\mathrm{e}^{\mathrm{i}\alpha \mathrm{X}} + \mathrm{O}(\epsilon^2).$$

By solving the linearised system, find $\widehat{A}$, and hence show that the reflected wave may be written as

$$\overline{p}(X, \overline{y}) = \epsilon\, r\, \mathrm{e}^{\mathrm{i}\alpha[\mathrm{X} - (\mathrm{M}^2-1)^{1/2}\overline{\mathrm{y}}]} + \text{c.c.}$$

with (the reflection coefficient)

$$r = \frac{(\mathrm{i}\alpha)^{2/3}(\mathrm{M}^2 - 1)^{1/2}\mathrm{Ai}'(0) + \alpha^2 \int_0^\infty \mathrm{Ai}(\zeta)\mathrm{d}\zeta}{(\mathrm{i}\alpha)^{2/3}(\mathrm{M}^2 - 1)^{1/2}\mathrm{Ai}'(0) - \alpha^2 \int_0^\infty \mathrm{Ai}(\zeta)\mathrm{d}\zeta} \ .$$

5. The stability of a three-dimensional boundary layer is studied by introducing a three-dimensional disturbance, and the perturbed flow field is written as

$$(u,\,v,\,w,\,p) = \Big(U_0(x/R,y),\,R^{-1}V_0(x/R,y), W_0(x/R,y),\,0\Big) + \epsilon(\widetilde{u},\,\widetilde{v},\,\widetilde{w},\,\widetilde{p}),$$

where x, y and z are non-dimensionalised by a reference boundary layer thickness δ, the Reynolds number R is based on δ, and $\epsilon \ll 1$ represents the magnitude of the perturbation.

Suppose that (u, v, w, p) satisfy the Navier-Stokes equations

$$\left.\begin{aligned} u_x + v_y + w_z &= 0, \\ uu_x + vu_y + wu_z &= -p_x + \tfrac{1}{R}(u_{xx} + u_{yy} + u_{zz}), \\ uv_x + vv_y + wv_z &= -p_y + \tfrac{1}{R}(v_{xx} + v_{yy} + v_{zz}), \\ uw_x + vw_y + ww_z &= -p_z + \tfrac{1}{R}(w_{xx} + w_{yy} + w_{zz}). \end{aligned}\right\}$$

(i) Derive the linearised equations governing the perturbation $(\widetilde{u},\,\widetilde{v},\,\widetilde{w},\,\widetilde{p})$. Indicate the terms which represent the non-parallel-flow effect.

(ii) Explain what is meant by Prandtl's parallel-flow approximation.

Suppose that this approximation is employed to seek a normal-mode solution of the form

$$(\widetilde{u},\,\widetilde{v},\,\widetilde{w},\,\widetilde{p}) = \Big(\widehat{u}(y),\,\widehat{v}(y),\,\widehat{w}(y),\widehat{p}(y)\Big)\mathrm{e}^{\mathrm{i}\{\alpha\mathrm{x}+\beta\mathrm{z}\}} + \mathrm{c.c.}$$

Derive the equations governing $\widehat{u}$, $\widehat{v}$, $\widehat{w}$ and $\widehat{p}$, and hence show that $\widehat{v}$ satisfies

$$\Big\{(U_0 + \frac{\beta}{\alpha}W_0)\Big(\frac{\partial^2}{\partial y^2} - \alpha^2\Big) - (U_{0,yy} + \frac{\beta}{\alpha}W_{0,yy}) - (\mathrm{i}\alpha\mathrm{R})^{-1}\Big(\frac{\partial^2}{\partial \mathrm{y}^2} - \alpha^2\Big)^2\Big\}\,\widehat{\mathrm{v}} = 0.$$

(iii) For $\alpha = O(1)$ and $\beta = O(1)$, what is the consistent governing equation in the limit $R \gg O(1)$? What is the order of error caused by neglecting the non-parallel flow effect?

(iv) Under the assumptions that $U_0 \sim \lambda_1 y$, $W_0 \sim \lambda_3 y$ as $y \to 0$, and $U_{0,yy} \neq 0$ and $W_{0,yy} \neq 0$ at $y = 0$, the base flow may be unstable to long-wavelength disturbances for which

$$\beta = R^{-3/7}\beta_0, \quad \alpha = -R^{-3/7}(\lambda_3/\lambda_1)\beta_0\,.$$

Show that viscosity is a leading-order effect in a sublayer where $y \sim O(R^{-1/7})$.