1. For the singular perturbation problem

$$\epsilon y'' + y' + y^2 = 0 \quad (0 < \epsilon \ll 1)\ , \quad y(0) = 2, \quad y(1) = \tfrac{1}{2},$$

seek an outer expansion

$$y(x) = y_0(x) + \epsilon y_1(x) + \dots\ .$$

Explain briefly why the inner (boundary) layer must be at $x = 0$, and hence determine the functions $y_0(x)$ and $y_1(x)$.

Seek an inner expansion

$$y(x) = Y_0(X) + \epsilon Y_1(X) + \dots\ ,$$

and calculate the first two terms, where $X = x/\epsilon$.

Use the two-term outer and inner expansions to construct a composite solution $C_{11}y$ based on the additive rule.

2. A three-dimensional flow past a flat plate of length L is described in a coordinate system (x, y, z) normalised by L, where x and z are on the plate surface with x and z being parallel and perpendicular to the leading edge respectively, while y is normal to the plate. The corresponding velocity components, normalised by a reference velocity U_∞, are denoted by (u, v, w), and the normalised pressure is denoted by p.

Assuming that the Reynolds number $R = U_\infty L/\nu >> 1$, where ν is the kinematic viscosity, write down the governing equations for the inviscid approximation (u_0, v_0, w_0, p_0), and specify appropriate boundary conditions.

By a brief scaling argument, deduce that the boundary layer lies in a layer $y \sim O(R^{-1/2})$, and that the expansion for (u, v, w, p) in the boundary layer takes the form

$$(u, v, w, p) = \Big(U(x, Y, z), R^{-1/2}V(x, Y, z), W(x, Y, z), P(x, z)\Big) + \ldots,$$

where $Y = R^{1/2}y$. Derive the boundary-layer equations, specify the appropriate boundary and matching conditions, and determine the pressure gradients P_x and P_z in terms of relevant slip velocities.

If the inviscid slip velocities are

$$u_0 = x^m, \quad w_0 = \beta \text{ (constant)},$$

there exists a similarity solution of the form

$$U = x^m f'(\eta), \quad V = -\tfrac{1}{2}x^{(m-1)/2}\Big[(m+1)f + (m-1)\eta f'(\eta)\Big], \quad W = g(\eta),$$

where $\eta = Y/s(x)$.

Deduce that $s(x) = x^{(1-m)/2}$, and show that f and g satisfy the equations

$$f''' + \tfrac{1}{2}(m+1)ff'' + m(1 - f'^2) = 0, \quad g'' + \tfrac{1}{2}(m+1)fg' = 0,$$

and state the boundary and matching conditions.

For the special case $m = 0$, deduce a simple relation between f and g.

3. In a suitably non-dimensionalised coordinate system (x, y), a steady flow is described by the Navier-Stokes equations

$$\left.\begin{aligned} u_x + v_y &= 0 \\ uu_x + vu_y &= -p_x + R^{-1}(u_{xx} + u_{yy}) \\ uv_x + vv_y &= -p_y + R^{-1}(v_{xx} + v_{yy}) \end{aligned}\right\} .$$

Suppose that the surface pressure gradient is such that in the vicinity of a point x_s, the boundary layer velocity field, $\Big(U_0(x, Y), R^{-\frac{1}{2}}V_0(x, Y)\Big)$, has the property that

$$U_0(x, Y) \to Y^2 \quad \text{as} \quad Y \to 0; \quad U_0(x, Y) \to 1 \quad \text{as} \quad Y \to \infty,$$

where $Y = R^{\frac{1}{2}}y$.

When the flow is perturbed by a sudden localised perturbation in the region $x - x_s = O(D) \ll 1$, the solution in the main part of the boundary layer can be sought in the form

$$u = U_0 + \epsilon U_1(X, Y) + \dots\ , \quad v = R^{-\frac{1}{2}}V_0 + \frac{\epsilon R^{-\frac{1}{2}}}{D}V_1(X, Y) + \dots\ ,$$

where $\epsilon \ll 1$, and $X = (x - x_s)/D$.

Derive the equations for U_1 and V_1, and find the solution for U_1 and V_1.

Examine the behaviour of V_1 as $Y \to \infty$, and explain why an upper deck is needed. Estimate the order of magnitude of the pressure in the upper deck.

Examine the behaviour of U_1 as $Y \to 0$, and deduce the width of this layer in terms of D and R. Estimate the streamwise velocity of the perturbation and the inertia in terms of D and ϵ.

Show that the flow becomes interactive when

$$D = O(R^{-2/7})\ .$$

Comment briefly on the role of each of the three decks.

Assuming that ϵ is sufficiently small, deduce the expansion for the flow field in the lower-deck and derive the governing equations, and specify the matching condition with the main-deck solution.

4. The boundary-layer flow of a stratified fluid subject to a localised unsteady suction may be described by triple deck theory, in which the lower deck is governed by the equations

$$\left.\begin{array}{rl} u_X + v_y = 0, & u_t + uu_X + vu_y = -P_X + u_{yy} \\ u = 0, \quad v = v_s(X,t) & \text{on} \quad y = 0 \\ u \to y + A(X,t) & \text{as} \quad y \to \infty \end{array}\right\} .$$

The pressure P is related to the upper deck pressure p via

$$P = p(X, 0, t).$$

In the upper deck, the pressure $p(X, \overline{y}, t)$ satisfies

$$\frac{\partial^2 p}{\partial X^2} + \frac{\partial^2 p}{\partial \overline{y}^2} + \gamma^2 p = 0,$$

subject to the boundary condition

$$\frac{\partial p}{\partial \overline{y}} = A_{XX} + \gamma^2 A \quad \text{on} \quad \overline{y} = 0; \qquad p \text{ is finite as } \overline{y} \to \infty \ ,$$

where $\gamma > 0$ is a constant representing the effect of stratification.

For a weak suction $v_s = HV_s(X)\,\mathrm{e}^{-\mathrm{i}\,\omega t} + c.c.$ with $H \ll 1$, seek a solution of the form

$$(u, v, P, A) = (y, 0, 0, 0) + H\Big(u_1(y), v_1(y), P_1, A_1\Big)\,\mathrm{e}^{-\mathrm{i}\,\omega t} + O(H^2).$$

Derive the linearised system including the boundary and matching conditions, and show that

$$-\mathrm{i}\,\omega u_{1,y} + y u_{1,Xy} - u_{1,yyy} = 0, \qquad u_{1,yy}(X, 0) = P_{1,X} + V_s(X).$$

Suppose that $\widehat{P}_1$ is the Fourier transform of P_1, i.e.

$$\widehat{P}_1 = \int_{-\infty}^{\infty} P_1(X)\,\mathrm{e}^{-\mathrm{i}\,kX}\,dX.$$

Solve the system by Fourier transform to show that

$$\widehat{P}_1 = -\frac{\widehat{V}_s(k)}{\mathrm{i}\,k\Delta(k,\omega)} \int_{\zeta_0}^{\infty} Ai(\zeta)\,d\zeta \quad \text{with} \quad \zeta = (\mathrm{i}\,k)^{\frac{1}{3}} y + \zeta_0, \quad \zeta_0 = -\mathrm{i}\,\omega(\mathrm{i}\,k)^{-2/3}$$

where $\widehat{V}_s$ is the Fourier transform of V_s, and

$$\Delta(k,\omega) = \int_{\zeta_0}^{\infty} Ai(\zeta)\,d\zeta + \mathrm{i}(\mathrm{i}\,k)^{2/3} Ai'(\zeta_0) / \Big(k(k^2 - \gamma^2)^{1/2}\Big),$$

with Ai being Airy function.

5. The stability of a two-dimensional boundary layer is studied by introducing a two-dimensional disturbance, and the perturbed flow field is written as

$$(u,\, v,\, p) = \Big(U_0(x/R, y),\; R^{-1}V_0(x/R, y),\; 0\Big) + \epsilon(\tilde{u},\, \tilde{v},\, \tilde{p})\ ,$$

where x, y are non-dimensionalised by a reference boundary layer thickness δ, the Reynolds number R is based on δ, and $\epsilon \ll 1$ represents the magnitude of the perturbation.

Given that (u, v, p) satisfy the Navier-Stokes equations

$$\left.\begin{aligned} u_x + v_y &= 0 \\ u_t + uu_x + vu_y &= -p_x + \tfrac{1}{R}(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y &= -p_y + \tfrac{1}{R}(v_{xx} + v_{yy}) \end{aligned}\right\} ,$$

derive the linearised equations governing the perturbation $(\tilde{u},\, \tilde{v},\, \tilde{p})$. Indicate the terms which represent the non-parallel-flow effect.

Explain what is meant by Prandtl's parallel-flow approximation.

Suppose that this approximation is employed to seek a normal-mode solution of the form

$$(\tilde{u},\, \tilde{v},\, \tilde{p}) = \Big(\widehat{u}(y),\, \widehat{v}(y),\, \widehat{p}(y)\Big)\, \mathrm{e}^{\mathrm{i}\{\alpha x - \omega t\}} + c.c..$$

Derive the equations governing $\widehat{u}$, $\widehat{v}$, $\widehat{p}$, and hence show that $\widehat{v}$ satisfies

$$\Big\{(U_0 - \omega/\alpha)\Big(\frac{\partial^2}{\partial y^2} - \alpha^2\Big) - U_{0,yy} - (\mathrm{i}\,\alpha R)^{-1}\Big(\frac{\partial^2}{\partial y^2} - \alpha^2\Big)^2\Big\}\widehat{v} = 0.$$

Comment on the appropriateness of this approach for explaining boundary-layer instability, in the following two cases:

(1) U_0 has an inflection point;

(2) U_0 has no inflection point.

In the case (2) above, relevant normal modes are of long wavelength in the sense that

$$\alpha \sim O(R^{-1/4}), \quad \omega \sim O(R^{-1/2}).$$

Show that viscosity is negligible for $y \sim O(1)$. Assuming that $U_0 \sim y$ as $y \to 0$, deduce that viscosity is a leading-order effect in a sublayer where $y \sim O(R^{-1/4})$. Show that in the $x-$ momentum equation for the perturbation, the ratio of the non-parallel-flow term to the viscous term is $O(R^{-3/4})$.