1. When fluid with viscosity μ_1 is forced at high pressure $p(x, y)$ through a narrow crack in a fuel tank its velocity $\mathbf{u}(x, y, z)$ obeys the Hele-Shaw equations

$$\nabla \cdot \mathbf{u} = 0, \qquad \mu_1 \frac{\partial^2 \mathbf{u}}{\partial z^2} = \nabla p \quad \text{in } 0 < z < a,$$

with $\mathbf{u} = 0$ on $z = 0$ and on $z = a$. Show that the z-averaged flow in the gap is given by

$$\overline{\mathbf{u}} \equiv \frac{1}{a} \int_0^a \mathbf{u}\, dz = -\frac{a^2}{12\mu_1} \nabla p.$$

From now on, assume that z-variation can be ignored and the flow is given by $\overline{\mathbf{u}}$.

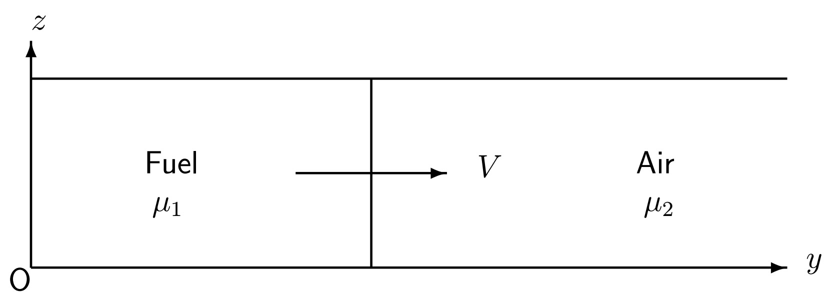
As the fuel escapes, it displaces air which obeys a similar equation, but with a different viscosity, μ_2. The interface between the two fluids is at $y = Vt$ and advances with constant velocity $(0, V, 0)$. The air is in $y > Vt$ and the fuel in $y < Vt$ as in the figure. Gravity and surface tension are negligible. The pressure on the interface is $p_i(t)$.

The interface is perturbed to $y = Vt + \varepsilon h_0 \zeta$, where $\zeta = e^{ikx+st}$ for constants ε, h_0, k and s, with $0 < \varepsilon \ll 1$ and $k > 0$. Writing $p = p_0(y,t) + \varepsilon p_1(y,t)\zeta$ in the air and $p = p_0(y,t) + \varepsilon p_2(y,t)\zeta$ in the fuel, find the functional forms of p_0, p_1 and p_2.

Show that the growth rate s is given by

$$s = kV \, \frac{(\mu_2 - \mu_1)}{(\mu_1 + \mu_2)}.$$

Discuss whether or not the interface is stable.



2. Show that the unidirectional flow $\mathbf{u} = (U(y), 0, 0)$ in $-1 < y < 1$ satisfies the inviscid equations of motion for any flow profile $U(y)$.

Perturbing the flow appropriately and assuming $U(y)$ is a smooth function, derive the Rayleigh equation

$$(U - c)(\psi'' - k^2\psi) + U''\psi = 0$$

defining $\psi(y)$, k and c and explaining when the flow is stable.

If $y = -1$ and $y = 1$ are rigid boundaries, what boundary conditions should ψ satisfy?

State without proof a sufficient condition for the flow to be stable to inviscid disturbances.

Suppose now that $U(y)$ is continuous but that its derivative is discontinuous at a single point $y = y_0$, with $-1 < y_0 < 1$. How should the problem for ψ be modified?

Determine whether or not the triangular profile $U(y) = 1 - |y|$ is stable.

3. (a) Summarise the advantages and disadvantages of using the Orr-Sommerfeld equation and the e^n-method for analysing boundary layer stability and transition. **[32%]**

(b) Explain carefully what Squire's theorem implies for the stability of plane parallel flows to three-dimensional disturbances for both viscous and inviscid flows. **[17%]**

(c) When a glass of water is instantaneously inverted, why does the water fall out? Discuss the preferred azimuthal and radial wavenumbers (m and k respectively). **[17%]**

(d) In terms of cylindrical polar coordinates (r, θ, z), the circular flow, $\mathbf{u} = (0, V(r), 0)$ in $R_1 < r < R_2$ has vorticity $\boldsymbol{\omega} = (0, 0, Q)$ where $Q = (rV)'/r$. Given that for all r in this range

$$\frac{dQ}{dr} < 0 \qquad \text{and} \qquad \frac{d}{dr}(r^2V^2) < 0,$$

what can be said about the stability of the flow? **[17%]**

(e) A uniform circular jet is surrounded by stationary fluid. What is the primary cause of instability when (i) the fluids are of equal density and (ii) the outer fluid is very much less dense?

In each case estimate roughly the most unstable wavelength in terms of the jet diameter a and speed V and other physical properties of the fluids. **[17%]**

4. (a) Explain the difference between convective and absolute instability.

The perturbation to a given flow obeys the equation

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + (R - R_c)u$$

where V, R and R_c are constants. Find the dispersion relation and show that the flow is stable if $R < R_c$.

If $R > R_c$, for which values of V is the flow absolutely unstable, and for which is it convectively unstable? **[40%]**

(b) Describe clearly, but in not too much detail, how to reduce the Orr-Sommerfeld equation

$$(U - c)(\psi'' - k^2\psi) - \psi U'' = \frac{1}{ikR_e}\left(\psi'''' - 2k^2\psi'' + k^4\psi\right)$$

with the boundary conditions

$$\psi(y_1) = 0 = \psi(y_2), \quad \psi'(y_1) = 0 = \psi'(y_2)$$

to the matrix eigenvalue problem

$$A\mathbf{x} + cB\mathbf{x} = 0,$$

where A and B are complex $N \times N$ matrices and $\mathbf{x}$ is an N-vector for a suitable integer N, using **either** a spectral method **or** finite differences. **[50%]**

(c) Sketch in the (k, R_e)-plane the neutral stability curves you would expect to obtain for the flows (i) $U = 1 - y^2$ and (ii) $U = \sin^2 \pi y$ in $-1 < y < 1$. **[10%]**